# Homework Set #7

**MasterMath: Set Theory**
*2019/20: 1st Semester*
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*Reminder.* From Homework Set #4 onwards, please hand in in ***collaboration teams*** of two students. These teams should work together, writing a joint solution to all of the exercises: both members of a team are fully responsible for all parts of the solution.

**Deadline for Homework Set #7:** Monday, 28 October 2019, 2pm.

Questions (32) and (33) provide a proof of the Cantor-Schröder-Bernstein Theorem without using the Axiom of Choice, so please avoid the use of AC in your answers. The *Cantor-Schröder-Bernstein Theorem* states (using the notation introduced in class): if $X \preccurlyeq Y$ and $Y \preccurlyeq X$, then $X \sim Y$.

(30) If $(P, \leq)$ is a partial order, we call $C \subseteq P$ a *chain* if $(C, \leq)$ is a total order. If $X \subseteq P$, we say that $X$ has an *upper bound* if there is some $p \in P$ such that for all $x \in X$, $x \leq p$. The partial order $(P, \leq)$ is called *chain-complete* if every chain has an upper bound. We say that $m \in P$ is a *maximal element* if there is no $p \in P$ such that $m < p$.

The statement "every non-empty chain-complete partial order has a maximal element" is called *Zorn's Lemma*. Show that on the basis of the axioms of ZF, Zorn's Lemma is equivalent to the Axiom of Choice.

(31) The following statement is known as *Hessenberg's Theorem*: for every $\alpha$, $\aleph_\alpha \times \aleph_\alpha \sim \aleph_\alpha$. Hessenberg's Theorem can be proved in ZF by constructing a bijection. In this question, you are going to give an alternative proof using Zorn's Lemma (cf. (30)).

Show the claim of Hessenberg's Theorem by induction on $\alpha$: assume that it is true for all $\gamma < \alpha$ and consider the set $H := \{f\,;$ there is an infinite subset $Z$ of $\aleph_\alpha$ such that $f$ is a bijection between $Z \times Z$ and $Z\}$. Order the set $H$ by inclusion and show that $(H, \subseteq)$ is a chain-complete partial order. Apply Zorn's Lemma to $(H, \subseteq)$ to obtain a maximal element $m : Z_m \times Z_m \to Z_m$. Show that $Z_m \sim \aleph_\alpha$ (*Hint.* Use the induction hypothesis!) and derive the theorem from that.

(32) Prove the *Knaster-Tarski Fixed Point Theorem*: Let $X$ be a set and $F : \wp(X) \to \wp(X)$ a $\subseteq$-monotone function, i.e., if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then $F$ has a fixed point, i.e., a set $A \subseteq X$ such that $A = F(A)$.

(33) Prove the *Banach Decomposition Theorem*: Let $X$ and $Y$ be sets and $f : X \to Y$ and $g : Y \to X$ arbitrary functions. Then there are disjoint decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $g[Y_2] = X_2$. Derive the Cantor-Schröder-Bernstein Theorem from the Banach Decomposition Theorem.

(*Hint.* Define $F(S) := X \backslash g[Y \backslash f[S]]$ and apply Knaster-Tarski.)