Homework Set #6

MasterMath: Set Theory
2019/20: 1st Semester
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Reminder. From Homework Set #4 onwards, please hand in in ***collaboration teams*** of two students. These teams should work together, writing a joint solution to all of the exercises: both members of a team are fully responsible for all parts of the solution.

Deadline for Homework Set #6: Monday, 21 October 2019, 2pm.

(25) In analogy to ordinal addition and multiplication, let us define *ordinal exponentiation* by recursion:

$$\begin{aligned} \alpha^0 &:= 1, \\ \alpha^{\beta+1} &:= \alpha^\beta \cdot \alpha, \\ \alpha^\lambda &:= \bigcup_{\xi<\lambda} \alpha^\xi \text{ (if } \lambda \text{ is a limit ordinal).} \end{aligned}$$

Check that this is an operation that is not commutative and associative.

Why is it not associative (even though the definition appears to be structurally very similar to that of $+$ and $\cdot$)?

Show that if $\alpha > 1$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$ and $\beta^\alpha \leq \gamma^\alpha$.

Give necessary and sufficient conditions on α and β so that α^β is a limit ordinal.

(26) Let γ be an ordinal. A finite sequence $(\gamma_0, ..., \gamma_n)$ with $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_n$ is called a *Cantor Normal Form of* γ if

$$\gamma = \omega^{\gamma_0} + ... + \omega^{\gamma_n}.$$

Prove that every ordinal $\gamma > 0$ has a unique Cantor Normal Form.

(27) Work in ZF and define for each set x its *Mirimanoff rank* by

$$\varrho(x) := \min\{\alpha \,;\, x \in \mathbf{V}_{\alpha+1}\}.$$

Prove that

(a) $x \in \mathbf{V}_\alpha$ if and only if $\varrho(x) < \alpha$,

(b) if $x \in y$, then $\varrho(x) < \varrho(y)$, and

(c) for all x, $\varrho(x) = \bigcup\{\varrho(y) + 1 \,;\, y \in x\}$.

(28) Consider the set $\mathbf{V}_{\omega+\omega}$, the $\omega + \omega$th level of the *von Neumann hierarchy.* Check which of the axioms of Zermelo set theory hold in the structure $(\mathbf{V}_{\omega+\omega}, \in)$.

In class, we showed that $\omega + \omega \notin \mathbf{V}_{\omega+\omega}$. Show that the axiom scheme of Replacement does not hold by providing a well-order $(X, R) \in \mathbf{V}_{\omega+\omega}$ that is isomorphic to $(\omega+\omega, \in)$. Explain why this is enough to refute the axiom scheme of Replacement.

Can you refute the axiom scheme of Replacement similarly for $\mathbf{V}_{\omega_1}$ where ω_1 is the smallest uncountable ordinal?

(29) Let X be a set of pairwise disjoint non-empty sets, i.e., if $x, x' \in X$, then $x \neq \varnothing \neq x'$ and $x \cap x' = \varnothing$. We say that C is a *choice set for* X if for each $x \in X$, the set $x \cap C$ has exactly one element. The *Axiom of Choice Sets* says that every set of pairwise disjoint, non-empty sets has a choice set.

Show that (on the basis of the axioms of ZF), the Axiom of Choice and the Axiom of Choice Sets are equivalent.

Why can't you get rid of the requirement that the sets in X are pairwise disjoint?