Homework Set #5

MasterMath: Set Theory
2019/20: 1st Semester
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Reminder. From Homework Set #4 onwards, please form ***collaboration teams*** of two students. These teams should work together, writing a joint solution to all of the exercises; it is **not** the intention that the questions are split between two people and each of them does only part of the work. Instead, the members of the team should meet, discuss, and write the solutions up jointly. Both members of a team are fully responsible for all parts of the solution.

Deadline for Homework Set #5: Monday, 14 October 2018, 2pm.

(20) Our proof of Hartogs's Theorem was done in ZF: it uses the unique representation of wellorders by ordinals which in turn uses the general recursion theorem (cf. (15)) that can only be proved with the Axiom Scheme of Replacement. Formulate and prove a version of Hartogs's Theorem that expresses "for every set there is a wellorder that cannot be injected in X" in Zermelo set theory Z without the Axiom Scheme of Replacement.

(21) If α and β are ordinals, we let $\alpha \otimes \beta$ be the unique ordinal μ such that $(\mu, \in) \cong (\alpha, \in) \otimes (\beta, \in)$, using the order product from (14). This is called the *synthetic definition of ordinal multiplication.* Prove that for any two ordinals α and β, we have $\alpha \otimes \beta = \alpha \cdot \beta$.

[*Hint.* As in the proof of the equivalence of the synthetic and recursive definitions of ordinal addition in class, consider specifying the precise bijection between the set $\alpha \times \beta$ and the ordinal $\alpha \cdot \beta$ as part of the induction hypothesis.]

(22) Prove the following properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.
(b) If $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$.
(c) If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.
(d) If $\alpha < \beta$ and $\gamma \neq 0$, then $\gamma \cdot \alpha < \gamma \cdot \beta$.
(e) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(f) If $\alpha^2\beta^2 = \beta^2\alpha^2$, then $\alpha\beta = \beta\alpha$.

The strict versions of (a) and (c) do not hold in general: give counterexamples.

(23) An ordinal γ is called a *gamma number* if it is closed under ordinal addition, i.e., if $\xi, \eta \in \gamma$, then $\xi + \eta \in \gamma$; it is called a *delta number* if it is closed under ordinal multiplication, i.e., if $\xi, \eta \in \gamma$, then $\xi \cdot \eta \in \gamma$. (Alternative names: "additively indecomposable", "principal number of addition" or "multiplicatively indecomposable", "principal number of multiplication", respectively.) Prove:

(a) An ordinal γ is a gamma number if and only if for all $\xi < \gamma$, we have $\xi + \gamma = \gamma$.

(b) An ordinal δ is a delta number if and only if for all $\xi < \delta$, we have $\xi \cdot \delta = \delta$.

Show furthermore that infinite initial ordinals are both gamma and delta numbers.

(24) An ordinal operation $F : \mathrm{Ord} \to \mathrm{Ord}$ is called *normal* if

(a) for all α, β, if $\alpha < \beta$, then $F(\alpha) < F(\beta)$, i.e., it is order-preserving, and

(b) for all limit ordinals λ, $F(\lambda) = \bigcup\{F(\xi)\,;\, \xi \in \lambda\}$ (this property is called *continuity*).

Show that every normal ordinal operation has arbitrarily large fixed points, i.e., α such that $F(\alpha) = \alpha$.