Homework Set #4

MasterMath: Set Theory
2019/20: 1st Semester
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From Homework Set #4 onwards, please form ***collaboration teams*** of two students. These teams should work together, writing a joint solution to all of the exercises; it is **not** the intention that the questions are split between two people and each of them does only part of the work. Instead, the members of the team should meet, discuss, and write the solutions up jointly. Both members of a team are fully responsible for all parts of the solution.

Deadline for Homework Set #4: Monday, 7 October 2018, 2pm.

(15) Let X be a set and R a binary relation on X. Define $R^{-1} := \{(y,x)\,;\,(x,y) \in R\}$. Show that if both R and R^{-1} are wellorders on X, then X must be finite.

(16) Let $(W,<)$ be a wellorder. For $w \in W$, we let $\mathrm{IS}(w) := \{x \in W\,;\, x < w\}$. Let Ψ be a functional and total formula in two free variables, i.e., if $\Psi(x,y)$ and $\Psi(x,y')$, then $y = y'$ and for all x there is an y such that $\Psi(x,y)$. Show the following version of the Recursion Theorem:

There is a unique function G with $\mathrm{dom}(G) = W$ such that for all $w \in W$, we have

$$G(w) = z \text{ if and only if } \Psi(G{\restriction}\mathrm{IS}(w), z).$$

(17) In class, we proved the uniqueness of ordinal representations for wellorders, i.e., $(X,<) \simeq (Y,<)$ if and only if $\mathrm{Eps}(X,<) = \mathrm{Eps}(Y,<)$. Our proof was not very detailed: go through the proof in detail and understand where the assumption that there is an *isomorphism* between X and Y is really used.

(18) Let α and β be ordinals. Show that if $\alpha \in \beta$, then $\mathrm{S}(\alpha) \in \mathrm{S}(\beta)$. Conclude that if $\alpha \neq \beta$, then $\mathrm{S}(\alpha) \neq \mathrm{S}(\beta)$.

(19) Let S be a set of ordinals. Show that exactly one of the following two cases occurs:

Case 1. S has no largest element, $\bigcup S \notin S$, and $\bigcup S$ is not the successor of any other ordinal;

Case 2. $\bigcup S$ is the largest element of S.