## Homework Set \#3

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Deadline for Homework Set \#3: Monday, 30 September 2019, 2pm.
(10) Prove the two distributive laws for arithmetic on $\mathbb{N}$ :

$$
\begin{aligned}
n \cdot(m+k) & =n \cdot m+n \cdot k \\
(n+m) \cdot k & =n \cdot k+m \cdot k
\end{aligned}
$$

Compare the two proofs: does one of them require more of the standard arithmetical properties of $\mathbb{N}$ than the other?
(11) Let $(X, \leq, 0, S)$ be a linear order with minimal element 0 and a unary $S: X \rightarrow X$ such that for all $x \in X$, we have $x<S(x)$.
(a) A subset $Z \subseteq X$ is called $S$-inductive if $0 \in Z$ and for all $x \in X$, if $x \in Z$, then $S(x) \in Z$.
(b) A subset $Z \subseteq X$ is called order inductive if for all $x \in X$, if $\{z \in X ; z<x\} \subseteq Z$, then $x \in Z$.
(c) We say that $(X, \leq, 0, S)$ satisfies the principle of complete induction if for every $S$ inductive set $Z$, we have that $Z=X$.
(d) We say that $(X, \leq, 0, S)$ satisfies the principle of order induction if for every order inductive set $Z$, we have that $Z=X$.
(e) We say that $(X, \leq, 0, S)$ satisfies the least number principle if every non-empty subset $Z \subseteq X$ has a $\leq$-least element.

Show that the principle of complete induction implies the principle of order induction and that the principle of order induction and the least number principle are equivalent. Give an example of a structure that satisfies the principle of order induction, but not the principle of complete induction. Give conditions on $S$ under which all three principles are equivalent.
(12) If $(T, \leq)$ is a linear order, we call a pair $(L, R)$ with $L, R \subseteq T$ a Dedekind cut if
(a) $L$ and $R$ partition $T$ (i.e., $L \cap R=\varnothing$ and $L \cup R=T$ ),
(b) for all $\ell \in L$ and all $r \in R$, we have $\ell<r$, and
(c) if $R$ has a smallest element, then $L$ has a largest element.

We write $\operatorname{Ded}(T, \leq)$ for the set of Dedekind cuts of $(T, \leq)$. We can order $\operatorname{Ded}(T, \leq)$ by $(L, R) \leq\left(L^{\prime}, R^{\prime}\right)$ if and only if $L \subseteq L^{\prime}$. Show that $(\operatorname{Ded}(T, \leq)$ is a linear order and that $(T, \leq)$ can be identified with a suborder of it. Check for a few examples what this order is (e.g., $T=\mathbb{N}, T=\mathbb{Z}, T=\{0,1,2,3\}, T=\mathbb{Q}, T=\operatorname{Ded}\left(T^{\prime}, \leq\right)$ for an arbitrary $\left(T^{\prime}, \leq\right)$, $T=\operatorname{Ded}(\mathbb{Q}, \leq) \backslash \mathbb{Q}$, or any other linear order you like).
A linear order $(L, \leq)$ is called complete if every subset bounded from above has a supremum. Show that $(\operatorname{Ded}(T, \leq), \leq)$ is complete.
(13) Consider the model $\mathcal{G}_{\infty}=\left(V_{\infty}, E_{\infty}\right)$ from the lectures and homework question (5) and check whether the axiom scheme of Replacement holds in it.
(14) If ( $X_{1}, \leq_{1}$ ) and ( $X_{2}, \leq_{2}$ ) are linear orders, we define the lexicographic order product

$$
\left(X_{1}, \leq_{1}\right) \otimes\left(X_{2}, \leq_{2}\right):=(P, \leq)
$$

as follows: $P:=X_{1} \times X_{2}$ and

$$
\left(x_{1}, x_{2}\right) \leq\left(x_{1}^{\prime}, x_{2}^{\prime}\right): \Longleftrightarrow x_{1}<x_{1}^{\prime} \vee\left(x_{1}=x_{1}^{\prime} \wedge x_{2} \leq x_{2}^{\prime}\right)
$$

Show that the lexicographic order product is a linear order and that it is a wellorder if and only if both ( $X_{1}, \leq_{1}$ ) and ( $X_{2}, \leq_{2}$ ) are.
(What happens if you define the order pointwise, i.e., $\left(x_{1}, x_{2}\right) \leq\left(x_{1}^{\prime}, x_{2}^{\prime}\right)$ if and only if $x_{1} \leq x_{1}^{\prime}$ and $x_{2} \leq x_{2}^{\prime}$ ?)

