Homework Set #9

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #9: Monday 12 November 2018, 2pm.

(39) Let $f : \omega_1 \to \mathbb{R}$ be an injective map. For $q \in \mathbb{Q}$ put $A_q = \{\alpha : f(\alpha) < q\}$ and $B_q = \{\alpha : f(\alpha) > q\}$. Let $I = \{q : A_q \text{ contains a cub set}\}$ and $J = \{q : B_q \text{ contains a cub set}\}$.

a. Prove: if $p \in I$ and $q \in J$ then $q < p$.
b. Prove: $I \neq \mathbb{Q}$ and $J \neq \mathbb{Q}$.
c. Prove: $\sup J < \inf I$ (by convention: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$).
d. Prove: there is a $q \in \mathbb{Q}$ such that both A_q and B_q are stationary.

(40) For every countable ordinal $\alpha \geqslant \omega$ let $f_\alpha : \alpha \to \omega$ be a bijection. For α and n define

$$U(\alpha, n) = \{\beta \in \omega_1 : \beta > \alpha \text{ and } f_\beta(\alpha) = n\}$$

Prove:

a. For every $n \in \omega$ the family $\{U(\alpha, n) : \alpha \geqslant \omega\}$ is pairwise disjoint.
b. For every $\alpha \geqslant \omega$ there is an n such that $U(\alpha, n)$ is stationary in ω_1.
c. There is an n such that $\{\alpha \geqslant \omega : U(\alpha, n) \text{ is stationary}\}$ is uncountable.
d. Every stationary subset of ω_1 can be decomposed into $\aleph_1$ many pairwise disjoint stationary sets. (Do not quote Solovay's theorem but give an alternative proof based on the considerations above.)

(41) Let $\{F_\alpha : \alpha < \omega_1\}$ be a family of finite subsets of ω_1. Prove that there are a finite set R and a stationary set S such that $F_\alpha \cap F_\beta = R$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$. *Hint*: Use Fodor's Pressing-Down Lemma to find R and a stationary set T such that $F_\alpha \cap \alpha = R$ for $\alpha \in T$.
(A family like $\{F_\alpha : \alpha \in S\}$ is called a Δ-system with root R.)

(42) As indicated in class identify $[\mathbb{R}]^2$ with $\{\langle x, y\rangle : x < y\}$.

a. Enumerate $\mathbb{Q}$ as $\langle q_n : n < \omega\rangle$. Define $F : [\mathbb{R}]^2 \to \omega$ by

$$F(\langle x, y\rangle) = \min\{n : x < q_n < y\}.$$

Show that there is no homogeneous triangle for F.

b. Let $\prec$ be a well-order of $\mathbb{R}$ and define $G : [\mathbb{R}]^2 \to 2$ by

$$G(\langle x, y\rangle) = \begin{cases} 0 & \text{if } y \prec x \\ 1 & \text{if } x \prec y \end{cases}$$

Show that there is no uncountable homogeneous set for G.