Homework Set #8

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #8: Monday 5 November 2018, 2pm.

(35) Remember that a set A was called *infinite* if it is not finite. A set A was called *finite* if there is $n \in \mathbb{N}$ such that $n \sim A$.
A set D if called *Dedekind infinite* if there is some proper subset X of D such that $X \sim D$. Show the following claims without the use of the Axiom of Choice:

a. Every Dedekind infinite set is infinite.

b. Every infinite set A with a choice function for $\wp(A)$ is Dedekind infinite.

c. The following are equivalent for a set D
 (1) D is Dedekind infinite
 (2) $\mathbb{N} \preccurlyeq D$
 (3) $D \sim D \cup \{P\}$ where P is some set not in D

d. A set A is infinite if and only if $\mathbb{N} \preccurlyeq \wp(\wp(A))$.

(36) Define an order $\prec$ on the pairs of ordinals as follows:

$$\langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle \qquad \text{iff} \qquad \begin{cases} \alpha + \beta < \gamma + \delta & \text{or} \\ \alpha + \beta = \gamma + \delta & \text{and } \alpha < \gamma \end{cases}$$

Prove, by induction, that for all cardinals infinite cardinals κ

a. $\kappa \times \kappa = \{\langle \alpha, \beta \rangle : \langle \alpha, \beta \rangle \prec \langle 0, \kappa \rangle\}$

b. The relation $\prec$ is a well-order of $\kappa \times \kappa$ and its order type is equal to κ.

(37) Let κ be a regular uncountable cardinal.

a. Let $n \in \mathbb{N}$ and let $f : \kappa^n \to \kappa$ be a function. Prove that $C_f = \{\alpha < \kappa : f[\alpha^n] \subseteq \alpha\}$ is closed and unbounded in κ.

b. Let $\lambda < \kappa$ be a cardinal and for each $\gamma < \lambda$ let $f_\gamma : \kappa^{n_\gamma} \to \kappa$ be a function, where $n_\gamma \in \mathbb{N}$. Prove that the set

$$C = \{\alpha < \kappa : (\forall \gamma < \lambda)(f_\gamma[\alpha^{n_\gamma}] \subseteq \alpha)\}$$

is closed and unbounded in κ.

c. Let $*$ be a binary operation on κ such that $\langle \kappa, * \rangle$ is a group. Prove that the set of $\alpha < \kappa$ for which $\langle \alpha, * \rangle$ is a subgroup is closed and unbounded in κ.

(38) Let κ be regular and uncountable and let $f : \kappa \to [\kappa]^{<\aleph_0}$ be a function (so $f(\alpha)$ is a finite subset of κ for all α).

a. Prove that $\{\alpha < \kappa : (\forall \beta < \alpha)(f(\beta) \subseteq \alpha)\}$ is closed and unbounded.

b. Prove that there is a $\gamma \in \kappa$ such that $\{\alpha < \kappa : \gamma = \max(f(\alpha) \cap \alpha)\}$ is stationary.

c. Prove that there is a stationary subset F of κ such that $\alpha \notin f(\beta)$ whenever α and β are distinct elements of F. (F is called a *free set* for f.)