Homework Set #7

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #7: Monday, 29 October 2018, 2pm.

Questions (31) and (32) provide a proof of the Cantor-Schröder-Bernstein Theorem without using the Axiom of Choice, so please avoid the use of AC in your answers. The *Cantor-Schröder-Bernstein Theorem* states (using the notation introduced in class): if $X \preccurlyeq Y$ and $Y \preccurlyeq X$, then $X \sim Y$.

(30) We write PSOWO ("power sets of ordinals are wellorderable") for the statement "if α is an ordinal, then $\wp(\alpha)$ is wellorderable". Prove in ZF that PSOWO implies AC.

(*Hint.* You may use the statement [not proved in the lecture] that every set is included in some $\mathbf{V}_\alpha$. Show that every set can be wellordered by transfinite induction on the Mirimanoff rank.)

(31) Prove the *Knaster-Tarski Fixed Point Theorem*: Let X be a set and $F : \wp(X) \to \wp(X)$ a $\subseteq$-monotone function, i.e., if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then F has a fixed point, i.e., a set $A \subseteq X$ such that $A = F(A)$.

(32) Prove the *Banach Decomposition Theorem*: Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ arbitrary functions. Then there are disjoint decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $g[Y_2] = X_2$. Derive the Cantor-Schröder-Bernstein Theorem from the Banach Decomposition Theorem.

(*Hint.* Define $F(S) := X \backslash g[Y \backslash f[S]]$ and apply Knaster-Tarski.)

(33) If X and Y are sets, we write $\mathrm{Fun}(X, Y)$ for the set of functions from X to Y. If κ and λ are cardinals, we define $\kappa + \lambda$, $\kappa \cdot \lambda$, and κ^λ as the unique cardinal that is in bijection with $\{0\} \times \kappa \cup \{1\} \times \lambda$, $\kappa \times \lambda$, and $\mathrm{Fun}(\lambda, \kappa)$, respectively.

Let κ, λ, and μ be cardinals. Show the following rules of cardinal arithmetic by providing concrete bijections between the appropriate sets:

(a) $(\kappa + \lambda) \cdot \mu = \kappa \cdot \mu + \lambda \cdot \mu$,

(b) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$, and

(c) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(34) A cardinal κ is called an *aleph fixed point* if $\aleph_\kappa = \kappa$. Show that for every ordinal α, there is a aleph fixed point $\kappa > \alpha$ with a countable cofinal subset $C \subseteq \kappa$. (Thus, κ is singular.)