Homework Set #6

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #6: Monday, 22 October 2018, 2pm.

(25) Prove the following properties of ordinal addition and multiplication:

(a) For all α and $\gamma \leq \gamma'$, we have that $\gamma + \alpha \leq \gamma' + \alpha$.
(b) For all α and $\gamma \leq \gamma'$, we have that $\gamma \cdot \alpha \leq \gamma' \cdot \alpha$.
(c) For all α, β, and γ, we have that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(d) If $\alpha \leq \beta$, then there is a unique pair (μ, ϱ) such that $\beta = \alpha \cdot \mu + \varrho$ and $\varrho < \alpha$.

(26) Prove the following *Synthetic Addition Theorem*: For any ordinals α and β, we have that $(\alpha + \beta, \in) \cong (\alpha, \in) \oplus (\beta, \in)$.

(27) Let γ be an ordinal. A finite sequence $(\gamma_0, ..., \gamma_n)$ with $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_n$ is called a *Cantor Normal Form of* γ if

$$\gamma = \omega^{\gamma_0} + ... + \omega^{\gamma_n}.$$

Prove that every ordinal $\gamma > 0$ has a unique Cantor Normal Form.

(28) Let X be a set of pairwise disjoint non-empty sets, i.e., if $x, x' \in X$, then $x \neq \varnothing \neq x'$ and $x \cap x' = \varnothing$. We say that C is a *choice set for* X if for each $x \in X$, the set $x \cap C$ has exactly one element. The *Axiom of Choice Sets* says that every set of pairwise disjoint, non-empty sets has a choice set. [Side remark: Why can't you get rid of the requirement that the sets in X are pairwise disjoint?]

Show that (on the basis of the axioms of ZF), the Axiom of Choice and the Axiom of Choice Sets are equivalent.

(29) Let $\mathbf{P} := (P, \leq)$ be a partial order. We say that $C \subseteq P$ is called a *chain in* $\mathbf{P}$ if the restricted order $(C, \leq \cap (C \times C))$ is a linear order. If $Z \subseteq P$ and $b \in P$, we say that b is an *upper bound of* Z if for all $z \in Z$, we have $z \leq b$. We say that m is a *maximal element of* $\mathbf{P}$ if for all $p \in P$, it is not the case that $m < p$. (Note that this does not necessarily imply that m is the *greatest element of* $\mathbf{P}$.) We say that $\mathbf{P}$ is *chain complete* if every chain in $\mathbf{P}$ has an upper bound. The following statement is known as *Zorn's Lemma* (ZL): every chain complete partial order has a maximal element.

Show that (on the basis of the axioms of ZF), the Axiom of Choice and Zorn's Lemma are equivalent.

Main ideas for both directions. For the direction "$\mathsf{AC} \Rightarrow \mathsf{ZL}$", assume that $\mathbf{P} = (P, \leq)$ does not have a maximal element, i.e., for each $p \in P$, the set $\{q \in P;\ p < q\}$ is non-empty, and use this assumption together with a choice function for $\wp(P)$ to get an injection from $\aleph(P)$ into P; for the direction "$\mathsf{ZL} \Rightarrow \mathsf{AC}$", consider the set of partial choice functions for X, i.e., functions f with $f(x) \in x$ and $\mathrm{dom}(f) \subseteq X$, ordered by inclusion and show that maximal elements in this partial order are choice functions.