Homework Set #5

MasterMath: Set Theory
2018/19: 1st Semester
K. P. Hart, Benedikt Löwe, & Robert Paßmann

Deadline for Homework Set #5: Monday, 15 October 2018, 2pm.

(20) A property P of sets is said to be *closed under binary unions* if whenever x and y have property P, then the set $x \cup y$ has property P. Consider the properties "x is a transitive set" and "$(x, \in)$ is transitive". Does one of them imply the other? (Give proofs or counterexamples.) Which one is closed under binary unions?

(21) If $(X, <)$ is a linear order and $I \subseteq X$, we called I an *initial segment* if for all $x, y \in X$, if $x < y$ and $y \in I$, then $x \in I$. An initial segment I was called *proper* if $I \neq X$. As before, for $x \in X$, $\mathrm{IS}(x) := \{y \in X\,;\, y < x\}$.

(a) Show that for arbitrary linear orders, $\mathrm{IS}(x)$ is a proper initial segment.

(b) Show that if $(X, <)$ is a wellorder, then I is a proper initial segment if and only if there is some $x \in X$ such that $I = \mathrm{IS}(x)$.

(22) Prove the *Rigidity Theorem for Wellorders*: Suppose $(X, <)$ and $(Y, <)$ are wellorders, I and J are initial segments of $(X, <)$ and $f : I \to Y$ and $g : J \to Y$ are isomorphisms. Show that $I = J$ and $f = g$.

(23) Prove the following claim we used in class: suppose that $(X, <)$ is a wellorder and that f is a function assigning an ordinal $f(x)$ to every element $x \in X$ such that $(f(x), \in) \cong (\mathrm{IS}(x), <)$. Show that f is order-preserving and that the range of f is a transitive set of ordinals.

(24) In the proof of Hartogs' Theorem, we considered a set X and the sets

$$A_X := \{(W, R)\,;\, W \subseteq X \text{ and } (W, R) \text{ is a wellorder}\} \text{ and}$$
$$B_X := \{\alpha\,;\, \exists (W, R) \in A \text{ such that } (W, R) \cong (\alpha, \in)\}.$$

We claimed that if X is infinite, then B_X is a transitive set of ordinals without largest elements. Prove that claim. Also, describe B_X in the case that X is finite.