Homework Set #4

MasterMath: Set Theory
2018/19: 1st Semester
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From Homework Set #4 onwards, if possible, please form ***collaboration teams*** of two students. These teams should work together, writing a joint solution to all of the exercises; it is **not** the intention that the questions are split between two people and each of them does only part of the work. Instead, the members of the team should meet, discuss, and write the solutions up jointly. Both members of a team are fully responsible for all parts of the solution.

Deadline for Homework Set #4: Monday, 8 October 2018, 2pm.

(15) Work in a model $\mathcal{G} = (V, \in)$ of Zermelo-Fraenkel set theory and do the following recursive construction:

$$\begin{aligned} x_0 &:= \varnothing, \\ x_{i+1} &:= \wp(x_i), \text{ and} \\ \mathbf{HF} &:= \bigcup\{x_i\,;\, i \in \mathbb{N}\}. \end{aligned}$$

On the set $\mathbf{HF}$, define an edge relation E by $x\,E\,y$ if and only if $x \in y$. Show that $(\mathbf{HF}, E)$ is a model of finite set theory.

(16) Work in a model $\mathcal{G} = (V, \in)$ of Zermelo-Fraenkel set theory and do the following recursive construction, building on the one in (15):

$$\begin{aligned} y_0 &:= \mathbf{HF}, \\ y_{i+1} &:= \wp(y_i), \text{ and} \\ M^* &:= \bigcup\{y_i\,;\, i \in \mathbb{N}\}. \end{aligned}$$

On M^*, define an edge relation E by $x\,E\,y$ if and only if $x \in y$. Show that (M^*, E) is a model of Zermelo set theory.

(17) Let $\mathcal{G} = (V, E)$ be a model of ZF_0. Assume that $\mathcal{G}$ satisfies $\in$-induction, i.e., for every formula Φ in $n+1$ free variables and all parameters $p_1, ..., p_n \in V$, if

$$\text{for all } x \in V, \text{ if for all } y\,E\,x, \text{ we have } \Phi(y, p_1, ..., p_n), \text{ then we have } \Phi(x, p_1, ..., p_n),$$

then for all $x \in V$, we have $\Phi(x, p_1, ..., p_n)$. Show that $\mathcal{G}$ is a model of ZF, i.e., satisfies the Axiom of Regularity.

(18) Work in ZF. A function s is called an *infinite decreasing chain* if $\mathrm{dom}(s) = \mathbb{N}$ and for all $i \in \mathbb{N}$, we have $s(i+1) \in s(i)$. Show that there are no infinite decreasing chains.

(19) Work in ZF. Let x be an ordinal, i.e., a transitive set such that $(x, \in)$ is a wellorder. Show that there is an ordinal y such that $(y, \in) \simeq (x, \in) \oplus (\mathbb{N}, <)$.

[*Remark on (18) & (19).* For your first exercises "working in ZF", it might be instructive to keep track of which axioms you use in the arguments, in particular the non-trivial ones such as Sep, Repl, and Reg.]