Homework Set #3

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #3: Monday, 1 October 2018, 2pm.

(10) A linear order $(L, \leq)$ is called *complete* if every subset bounded from above has a supremum. It is called *dense* if for any $x < y$, there is some z such that $x < z < y$. In class, we have seen the construction of the linear order of *Dedekind cuts* $\mathcal{D}(L, \leq)$ of a linear order. Show that if $(L, \leq)$ is complete, dense, and has neither largest nor smallest element, then $\mathcal{D}(L, \leq)$ is isomorphic to $(L, \leq)$.

(11) Consider the linear order $(\mathbb{Q}, \leq)$ of rational numbers. Show that for any rational numbers $p < q$ and any Dedekind cut (I, F) there are p', q' such that $p < p' < q' < q$ and either $p' \in F$ or $q' \in I$.

(12) Suppose that $\sigma : \mathbb{N} \to \mathcal{D}(\mathbb{Q}, \leq)$ is a function. Use (11) to construct sequences $(p_i\,; i \in \mathbb{N})$ and $(q_i\,; i \in \mathbb{N})$ such that $p_i < p_{i+1} < q_{i+1} < q_i$ and for $\sigma(i) = (I, F)$ we have either $p_{i+1} \in F$ or $q_{i+1} \in I$. Use these sequences to show that σ is not a surjection.

(13) Let $(X, \leq, 0, S)$ be a linear order with minimal element 0 and a unary, increasing function $S : X \to X$, i.e., for all $x \in X$, we have $x < S(x)$.

   (a) A subset $Z \subseteq X$ is called *S-inductive* if $0 \in Z$ and for all $x \in X$, if $x \in Z$, then $S(x) \in Z$.
   (b) A subset $Z \subseteq X$ is called *order inductive* if for all $x \in X$, if $\{z \in X\,; z < x\} \subseteq Z$, then $x \in Z$.
   (c) We say that $(X, \leq, 0, S)$ *satisfies the principle of complete induction* if for every S-inductive set Z, we have that $Z = X$.
   (d) We say that $(X, \leq, 0, S)$ *satisfies the principle of order induction* if for every order inductive set Z, we have that $Z = X$.

   Show that if $(X, \leq, 0, S)$ satisfies the principle of complete induction, it satisfies the principle of order induction, and give conditions on S under which the converse holds.

   [*Note.* We have seen in class that the converse does not always hold, since the order $(\mathbb{N}, \leq) \oplus (\mathbb{N}, \leq)$ satisfies the principle of order induction, but not the principle of complete induction.]

(14) Let $(W, <)$ be a wellorder. For $w \in W$, we let $\mathrm{IS}_w := \{x \in W\,; x < w\}$. Let Ψ be a functional and total formula in two free variables, i.e., if $\Psi(x, y)$ and $\Psi(x, y')$, then $y = y'$ and for all x there is an y such that $\Psi(x, y)$. Show the *4th Version of the Recursion Theorem*:

   There is a unique function g with $\mathrm{dom}(g) = W$ such that for all $w \in W$, we have

$$g(w) = z \text{ if and only if } \Psi(g{\upharpoonright}\mathrm{IS}_w, z).$$