Deadline for Homework Set #3: Monday, 1 October 2018, 2pm.

(10) A linear order \((L, \leq)\) is called complete if every subset bounded from above has a supremum. It is called dense if for any \(x < y\), there is some \(z\) such that \(x < z < y\). In class, we have seen the construction of the linear order of Dedekind cuts \(\mathcal{D}(L, \leq)\) of a linear order. Show that if \((L, \leq)\) is complete, dense, and has neither largest nor smallest element, then \(\mathcal{D}(L, \leq)\) is isomorphic to \((L, \leq)\).

(11) Consider the linear order \((\mathbb{Q}, \leq)\) of rational numbers. Show that for any rational numbers \(p < q\) and any Dedekind cut \((I, F)\) there are \(p', q'\) such that \(p < p' < q' < q\) and either \(p' \in F\) or \(q' \in I\).

(12) Suppose that \(\sigma : \mathbb{N} \to \mathcal{D}(\mathbb{Q}, \leq)\) is a function. Use (11) to construct sequences \((p_i; i \in \mathbb{N})\) and \((q_i; i \in \mathbb{N})\) such that \(p_i < p_{i+1} < q_{i+1} < q_i\) and for \(\sigma(i) = (I, F)\) we have either \(p_{i+1} \in F\) or \(q_{i+1} \in I\). Use these sequences to show that \(\sigma\) is not a surjection.

(13) Let \((X, \leq, 0, S)\) be a linear order with minimal element 0 and a unary, increasing function \(S : X \to X\), i.e., for all \(x \in X\), we have \(x < S(x)\).

(a) A subset \(Z \subseteq X\) is called \(S\)-inductive if \(0 \in Z\) and for all \(x \in X\), if \(x \in Z\), then \(S(x) \in Z\).

(b) A subset \(Z \subseteq X\) is called order inductive if for all \(x \in X\), if \(\{z \in X ; z < x\} \subseteq Z\), then \(x \in Z\).

(c) We say that \((X, \leq, 0, S)\) satisfies the principle of complete induction if for every \(S\)-inductive set \(Z\), we have that \(Z = X\).

(d) We say that \((X, \leq, 0, S)\) satisfies the principle of order induction if for every order inductive set \(Z\), we have that \(Z = X\).

Show that if \((X, \leq, 0, S)\) satisfies the principle of complete induction, it satisfies the principle of order induction, and give conditions on \(S\) under which the converse holds.

[Note. We have seen in class that the converse does not always hold, since the order \((\mathbb{N}, \leq) \oplus (\mathbb{N}, \leq)\) satisfies the principle of order induction, but not the principle of complete induction.]

(14) Let \((W, <)\) be a wellorder. For \(w \in W\), we let \(\text{IS}_w := \{x \in W ; x < w\}\). Let \(\Psi\) be a functional and total formula in two free variables, i.e., if \(\Psi(x, y)\) and \(\Psi(x, y')\), then \(y = y'\) and for all \(x\) there is an \(y\) such that \(\Psi(x, y)\). Show the 4th Version of the Recursion Theorem:

There is a unique function \(g\) with \(\text{dom}(g) = W\) such that for all \(w \in W\), we have

\[g(w) = z\text{ if and only if }\Psi(g|\text{IS}_w, z)\]