Homework Set #14

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #14: Monday 17 December 2018, 2pm.

(59) Define a relation Q on $\mathbb{R}$ by

$$x \, Q \, y \qquad \text{if} \qquad x - y \in \mathbb{Q}$$

a. Verify that Q is an equivalence relation.

b. Construct a perfect set P in $\mathbb{R}$ such that $x \not{Q} \, y$ whenever x and y are distinct members of P. *Hint*: Enumerate $\mathbb{Q} \setminus \{0\}$ as $\langle q_n : n \in \omega \rangle$. Build a family of closed intervals I_s, indexed by the tree $2^{<\omega}$ of finite sequences of 0s and 1s such that 1) $I_{\langle\rangle} = [0,1]$, 2) if $s \subset t$ then $I_s \supset I_t$, 3) for each n the family $\{I_s : s \in 2^n\}$ is pairwise disjoint, and 4) for each n: if $s, t \in 2^n$ and $s \neq t$ then $(I_s - q_n) \cap I_t = \emptyset$.

c. Apply the Axiom of Choice to find a subset V of $[0, \frac{1}{2}]$ that intersects each Q-equivalence class in exactly one point.

d. Prove that this V is not Lebesgue-measurable. *Hint*: $\{V + q : q \in \mathbb{Q}\}$ is a pairwise disjoint family of subsets of $\mathbb{R}$ that covers $\mathbb{R}$; also note that $\bigcup_{n=2}^{\infty}(V + \frac{1}{n}) \subseteq [0,1]$. These facts imply contradicting information about the possible measure of V.

e. Prove that the set V does not have the Baire property. *Hint*: Note that V cannot be meager. Assume V does have the Baire property and show that there is an interval (a,b) such that $(a,b) \setminus V$ is meager. Deduce that $(a,b) \cap (V + q)$ is meager for all $q \neq 0$ and hence $(a - q, b - q) \cap V$ is meager for all $q \neq 0$. Deduce that V is meager after all.

f. Show that V may be chosen so that it does have a perfect subset.

(60) Prove that there is a subset B of $\mathbb{R}$ such that neither B nor $\mathbb{R} \setminus B$ has a perfect subset. *Hint*: Enumerate the family of all perfect subsets of $\mathbb{R}$ as $\langle P_\alpha : \alpha < \mathfrak{c} \rangle$, where $\mathfrak{c} = |\mathbb{R}|$ (why is this possible?). By recursion construct injective sequences $\langle x_\alpha : \alpha < \mathfrak{c} \rangle$ and $\langle y_\alpha : \alpha < \mathfrak{c} \rangle$ with disjoint ranges such that $x_\alpha, y_\alpha \in P_\alpha$ for all α.