(54) Let $X$ be a Polish space, with topology $\tau$, and $F$ a closed subset of $X$. Let $\sigma$ be the smallest topology that contains $\tau \cup \{F\}$. Prove that $\sigma$ is a Polish topology and that the Borel sets defined from $\sigma$ are exactly the Borel sets defined from $\tau$.

(55) Let $\{A_n : n \in \omega\}$ be a pairwise disjoint family of analytic subsets of $X$. Prove that there is a pairwise disjoint family $\{C_n : n \in \omega\}$ of Borel sets such that $A_n \subseteq C_n$ for all $n$.

(56) Let $\{A_s : s \in \text{Seq}\}$ be a family of Borel sets such that $A_s \supseteq A_t$ whenever $s \subseteq t$ and such that $A_{s^{-}n} \cap A_{s^{-}m} = \emptyset$, whenever $s \in \text{Seq}$ and $n \neq m$. Prove that $\bigcup_{a \in \mathbb{N}} \bigcap_{n \in \omega} A_a|n$ is a Borel set. *Hint:* Try to interchange $\bigcup$ and $\bigcap$.

(57) Prove that there are a set of measure zero $Z$ and a meager set $M$ such that $\mathbb{R} = Z \cup M$. *Hint:* Inspect the proof in class that $\mu^*(\mathbb{Q}) = 0$.

(58) Let $r \in (0, 1)$. Construct a closed nowhere dense subset $D_r$ of $[0, 1]$ such that $\mu(D_r) = r$. 