Homework Set #11

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #11: Monday 26 November 2018, 2pm.

(46) On the 'smallest σ-algebra that contains $\mathcal{X}$'. Let S be a set and $\mathcal{X}$ a subfamily of $\mathcal{P}(X)$.

a. Show that $\mathcal{P}(X)$ and the family $\{\emptyset, S\}$ are σ-algebras on S.
b. Prove: if $\mathfrak{S}$ is a family of σ-algebras on S then $\bigcap \mathfrak{S}$ is a σ-algebra on S.
c. There is a σ-algebra $\mathcal{S}$ such that $\mathcal{X} \subseteq \mathcal{S}$ and whenever $\mathcal{B}$ is a σ-algebra with $\mathcal{X} \subseteq \mathcal{B}$ then $\mathcal{S} \subseteq \mathcal{B}$.
d. Let $\mathcal{X} = \{\{x\} : x \in S\}$; describe the smallest σ-algebra on S that contains $\mathcal{X}$.
e. Describe the Borel σ-algebra of the metric space $\mathbb{Q}$.

(47) Consider the following σ-algebras on $\mathbb{R}$.

- $\mathcal{B}_1$ is the σ-algebra of Borel sets
- $\mathcal{B}_2$ is the smallest σ-algebra that contains all open intervals with rational end points

Prove that these σ-algebras are identical.

(48) Application of the Baire Category Theorem.
Prove that $\mathbb{Q}$ is an F_σ-set but not a G_δ-set in $\mathbb{R}$.

(49) The space $\mathcal{N}$.

a. Prove that d, as defined in class:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \neq y \text{ and } n = \min\{k : x_k \neq y_k\} \end{cases}$$

is a metric on $\mathcal{N}$.

(50) Let $\langle q_n : n \in \omega \rangle$ be a one-to-one enumeration of $\mathbb{Q}$. Define $f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ as follows

$$\begin{aligned} f(q_n) &= q_{2n} + \sqrt{2} \\ f(q_n + \sqrt{2}) &= q_{2n+1} + \sqrt{2} \\ f(x) &= x \text{ in all other cases} \end{aligned}$$

a. Prove that f is a bijection.
b. Prove for all subsets B of $\mathbb{R}$ that B is a Borel set in $\mathbb{R}$ if and only if $f[B]$ is a Borel set in $\mathbb{R} \setminus \mathbb{Q}$. *Hint*: Show that $f[O]$ is Borel when O is open in $\mathbb{R}$ and that $f^{-1}[O]$ is Borel when O is open in $\mathbb{R} \setminus \mathbb{Q}$.