Homework Set #11

Deadline for Homework Set #11: Monday 26 November 2018, 2pm.

46) On the ‘smallest $\sigma$-algebra that contains $\mathcal{X}$’. Let $S$ be a set and $\mathcal{X}$ a subfamily of $\mathcal{P}(S)$.
   a. Show that $\mathcal{P}(S)$ and the family $\{\emptyset, S\}$ are $\sigma$-algebras on $S$.
   b. Prove: if $\mathcal{G}$ is a family of $\sigma$-algebras on $S$ then $\bigcap \mathcal{G}$ is a $\sigma$-algebra on $S$.
   c. There is a $\sigma$-algebra $\mathcal{S}$ such that $\mathcal{X} \subseteq \mathcal{S}$ and whenever $\mathcal{B}$ is a $\sigma$-algebra with $\mathcal{X} \subseteq \mathcal{B}$ then $\mathcal{S} \subseteq \mathcal{B}$.
   d. Let $\mathcal{X} = \{\{x\} : x \in S\}$; describe the smallest $\sigma$-algebra on $S$ that contains $\mathcal{X}$.
   e. Describe the Borel $\sigma$-algebra of the metric space $\mathbb{Q}$.

47) Consider the following $\sigma$-algebras on $\mathbb{R}$.
   - $\mathcal{B}_1$ is the $\sigma$-algebra of Borel sets
   - $\mathcal{B}_2$ is the smallest $\sigma$-algebra that contains all open intervals with rational end points

   Prove that these $\sigma$-algebras are identical.

48) Application of the Baire Category Theorem.
   Prove that $\mathbb{Q}$ is an $F_\sigma$-set but not a $G_\delta$-set in $\mathbb{R}$.

49) The space $\mathcal{N}$.
   a. Prove that $d$, as defined in class:
      
      \[ d(x, y) = \begin{cases} 
      0 & \text{if } x = y \\
      2^{-n} & \text{if } x \neq y \text{ and } n = \min\{k : x_k \neq y_k\} 
      \end{cases} \]

      is a metric on $\mathcal{N}$.

50) Let $\langle q_n : n \in \omega \rangle$ be a one-to-one enumeration of $\mathbb{Q}$. Define $f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ as follows
      
      \[
      f(q_n) = q_{2n} + \sqrt{2} \\
      f(q_n + \sqrt{2}) = q_{2n+1} + \sqrt{2} \\
      f(x) = x \text{ in all other cases}
      \]

   a. Prove that $f$ is a bijection.
   b. Prove for all subsets $B$ of $\mathbb{R}$ that $B$ is a Borel set in $\mathbb{R}$ if and only if $f[B]$ is a Borel set in $\mathbb{R} \setminus \mathbb{Q}$. \textit{Hint:} Show that $f[O]$ is Borel when $O$ is open in $\mathbb{R}$ and that $f^{-1}[O]$ is Borel when $O$ is open in $\mathbb{R} \setminus \mathbb{Q}$. 