Homework Set #10

MasterMath: Set Theory
2018/19: 1st Semester
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Deadline for Homework Set #10: Monday 19 November 2018, 2pm.

(43) Two applications of Ramsey's theorem.

a. Let $\langle L, < \rangle$ be an infinite linearly ordered set. Prove that L has an infinite subset X that is well-ordered by $<$ or an infinite subset Y that is well-ordered by $>$.

b. Let $\langle P, < \rangle$ be an infinite partially ordered set. Prove that P has an infinite subset C that is linearly ordered by $<$ or an infinite subset U that is unordered by $<$, which means that if x and y in U are distinct then neither $x < y$ nor $y < x$.

(44) An application of the theorem of Erdős, Dushnik and Miller. Let κ be an infinite cardinal number and let $\prec$ be some well-order of κ. Prove that there is a subset A of κ of cardinality κ and such that for all $\alpha, \beta \in A$ we have

$$\alpha \in \beta \text{ if and only if } \alpha \prec \beta.$$

(45) An ordinal strengthening of the Erdős-Dushnik-Miller theorem. The statement

$$\omega_1 \to (\omega_1, \omega + 1)^2$$

means: if $[\omega_1]^2 = A \cup B$ then there is subset H of ω_1 of cardinality $\aleph_1$ such that $[H]^2 \subseteq A$ or there is a subset I of ω_1 of *order-type* $\omega + 1$ such that $[I]^2 \subseteq B$.
We assume there is no B-homogeneous set of order-type $\omega + 1$.

a. Prove that for every limit ordinal α there is a maximal subset K_α of α such that $[K_\alpha \cup \{\alpha\}]^2 \subseteq B$.

b. Deduce that every K_α is finite.

c. Apply the Pressing-Down Lemma to find a stationary set S and one finite set K such that $K_\alpha = K$ for all $\alpha \in S$.

d. Prove that $[S]^2 \subseteq A$.