2017-11-20

8.6 $f : S \to \mathrm{ORD}$ IS REGRESSIVE

IF $(\forall \alpha \in S)(\alpha > 0 \to f(\alpha) < \alpha)$

8.7 FODOR – PRESSING-DOWN LEMMA

IF $S \subseteq \kappa$ IS STATIONARY AND $f : S \to \kappa$ IS REGRESSIVE THEN THERE IS A STATIONARY SET T ON WHICH f IS CONSTANT

□ Suppose not

So FOR EVERY $\gamma < \kappa$ THERE IS A CUB C_γ SUCH THAT $C_\gamma \cap \{\alpha \in S : f(\alpha) = \gamma\} = \emptyset$

LET $C = \Delta_{\gamma < \kappa} C_\gamma$ AND TAKE $\alpha \in S \cap C$ $(\alpha > 0)$

THEN $\alpha \in C_{f(\alpha)}$ HENCE $f(\alpha) \neq f(\alpha)$ ... □

κ REGULAR UNCOUNTABLE $\lambda < \kappa$ REGULAR

$$E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}\,\alpha = \lambda\}$$

THIS SET IS STATIONARY.

So $E^\kappa_{\omega_0}$ AND $E^\kappa_{\omega_1}$ ARE DISJOINT STATIONARY SETS

IF $\kappa \geq \aleph_2$. THE CUB FILTER IS NOT AN ULTRAFILTER

8.8 EVERY STATIONARY SUBSET OF E^κ_ω IS THE UNION OF κ DISJOINT STATIONARY SUBSETS

□ LET $W \subseteq E^\kappa_\omega$ BE STATIONARY

FOR EACH $\alpha \in W$ CHOOSE $\langle a(\alpha, n) : n < \omega \rangle$ INCREASING AND COFINAL IN α.

• THERE IS AN n SUCH THAT FOR ALL $\eta < \kappa$

$$W(n, \eta) = \{\alpha \in W : a(\alpha, n) \geq \eta\}$$

IS STATIONARY

IF NOT LET $\eta_n < \kappa$ AND C_n BE CUB SUCH THAT $C_n \cap W(n, \eta_n) = \emptyset$

2017 - 11 - 20

LET $\eta = \sup\{\eta_n : n < \omega\}$ AND $C = \bigcap_{n<\omega} C_n$.

THEN FOR $\alpha \in C \cap W$ WITH $\alpha > \eta$

$$a(\alpha, n) < \eta$$

FOR ALL n, SO ... $\sup_n a(\alpha, n) \leq \eta < \alpha$.

CONTRADICTION

TAKE SUCH AN n. AND CONSIDER $f : W \to \kappa$

DEFINED BY $f(\alpha) = a(\alpha, n)$.

FOR EVERY η THERE IS $\gamma_\eta \geq \eta$ SUCH

THAT $S_\eta = \{\alpha \in W : f(\alpha) = \gamma_\eta\}$

IS STATIONARY

DEFINE $\eta_0 = 0$

$\eta_{\alpha+1} = \gamma_\eta$ WHERE $\eta = \sup\{\eta_\delta + 1 : \delta < \alpha\}$

SO $\langle \eta_\alpha : \alpha < \kappa \rangle$ IS INCREASING

AND IT GIVES κ MANY DISJOINT

STATIONARY SETS. □

SAME PROOF WORKS FOR E^κ_λ

NEXT IF W IS A STATIONARY SUBSET OF $\{\alpha < \kappa : \mathrm{cf}\,\alpha < \alpha\}$

THEN THERE IS $\lambda < \kappa$ SUCH THAT $\{\alpha \in W : \mathrm{cf}\,\alpha = \lambda\}$

IS STATIONARY HENCE WE CAN SPLIT W.

SO WHAT TO DO IF $\{\alpha < \kappa : \mathrm{cf}\,\alpha = \alpha\}$ IS STATIONARY

8.9 LET $S \subseteq \kappa$ BE STATIONARY AND $\mathrm{cf}\,\alpha = \alpha$ FOR $\alpha \in S$

THEN $T = \{\alpha \in S : S \cap \alpha$ IS <u>NOT</u> STATIONARY$\}$ <u>IS</u> STATIONARY

□ LET C BE CUB; LET $C' = \{\alpha \in C : \alpha = \sup(C \cap \alpha)\}$

TAKE $\alpha = \min C' \cap S$ THEN α IS REGULAR AND

$C \cap \alpha$ IS CUB IN α AND HENCE SO IS $C' \cap \alpha$.

BUT $C' \cap S \cap \alpha = \emptyset$ SO $S \cap \alpha$ IS NOT STATIONARY IN α

$\therefore\ C \cap T \neq \emptyset$ □

2017-11-20

8.10 LET κ BE REGULAR AND UNCOUNTABLE.

THEN EVERY STAT SUBSET CAN BE PARTITIONED INTO κ MANY (DISJOINT) STATIONARY SETS.

□ LET $A \subseteq \kappa$ BE STATIONARY

- IF $\{\alpha \in A : \mathrm{cf}\,\alpha < \alpha\}$ IS STATIONARY THEN WE ARE DONE
- CONSIDER THE CASE $\{\alpha \in A : \mathrm{cf}\,\alpha = \alpha\}$ IS STAT.

BY 8.9 $W = \{\alpha \in A : \mathrm{cf}\,\alpha = \alpha \wedge A \cap \alpha \text{ NOT STAT IN } \alpha\}$ IS STATIONARY

WE WORK WITH W.

FOR $\alpha \in W$ CHOOSE $\langle a(\alpha, \xi) : \xi < \alpha \rangle$ CONTINUOUS, INCREASING, COFINAL IN α.

AND $\{a(\alpha,\xi) : \xi < \alpha\} \cap W = \emptyset$

AS BEFORE: THERE IS ONE ξ SUCH THAT FOR ALL $\eta < \kappa$ THE SET $W(\eta,\xi) = \{\alpha \in W : a(\alpha,\xi) \geq \eta\}$ IS STATIONARY

OTHERWISE: FOR EACH ξ FIND η_ξ AND C_ξ SUCH THAT $W(\eta_\xi, \xi) \cap C_\xi = \emptyset$

SO: $\alpha \in W \cap C_\xi \rightarrow a(\alpha,\xi) < \eta_\xi$

LET $C = \Delta_{\xi<\kappa} C_\xi$ AND $D = \{\gamma \in C : \xi < \gamma \rightarrow \eta_\xi < \gamma\}$

C IS CUB AND D IS CUB

LET $\gamma < \alpha$ IN $W \cap D$

IF $\xi < \gamma$ THEN $a(\alpha,\xi) < \eta_\xi < \gamma$

SO THAT $a(\alpha,\gamma) = \gamma \in W$

BUT $a(\alpha,\gamma) \notin W$

NOW FOLLOW THE PROOF OF LEMMA 8.8 ∎

COMBINATORIAL SET THEORY

PARTITION PROPERTIES.

A A SET, $m \in \mathbb{N}$, $m > 0$:

$$[A]^m = \{X \subseteq A : |X| = m\}$$

(SOMETIMES $[A]^m = \{s \in A^m : s \text{ IS INCREASING}\}$)

9.0a IF $f : [6]^2 \to 2$ THEN THERE ARE $a, b, c \in 6$ SUCH THAT f IS CONSTANT ON $[\{a,b,c\}]^2$.

PROOF LOOK AT 0

$$X_0 = \{i : f(\{0,i\}) = 0\}$$
$$X_1 = \{i : f(\{0,i\}) = 1\}$$

ONE OF THESE HAS AT LEAST THREE ELEMENTS

SAY X_ε

CASE 1 THERE ARE $a, b \in X_\varepsilon$ WITH $f(\{a,b\}) = \varepsilon$

THEN $\{a, b, 0\}$ IS AS REQUIRED

CASE 2 $f(\{a,b\}) = 1-\varepsilon$ FOR ALL $a, b \in X_\varepsilon$

THEN X_ε IS AS REQUIRED.

9.1 LET $n, k \in \mathbb{N}$:

FOR EVERY $F : [\omega]^n \to k$ THERE IS AN INFINITE $H \subseteq \omega$ SUCH THAT $F \restriction [H]^n$ IS CONSTANT

□ $n = 1$: $F : \omega \to k$ IS CONSTANT ON AN INFINITE SET

$n = 2$. LET $F : [\omega]^2 \to k$

FOR $a \in \omega$ DEFINE $F_a : [a+1, \omega) \to k$

BY $F_a(m) = F(\{a, m\})$

DEFINE $x_0 = 0$ AND TAKE X_0 INFINITE ON WHICH F_0 IS CONSTANT

GIVEN x_m AND X_m

LET $x_{m+1} = \min X_m$

AND TAKE $X_{m+1} \subseteq X_m$ INFINITE SUCH THAT $F_{x_{m+1}} \restriction X_{m+1}$ IS CONSTANT.

CONSIDER $X = \{x_m : m \in \omega\}$

NOTE:

IF $x_\ell < x_p < x_q$

THEN $F(\{x_\ell, x_p\}) = F_{x_\ell}(x_p) = F_{x_\ell}(x_q)$
$= F(\{x_\ell, x_q\})$

BECAUSE $x_p, x_q \in X_\ell$ X_k

LET $f(\ell)$ BE THE CONSTANT VALUE OF $F_{x_\ell} \restriction X_\ell$

TAKE I INFINITE SUCH THAT $f \restriction I$ IS CONSTANT

$H = \{x_i : i \in I\}$ IS HOMOGENEOUS

$F(\{x_i, x_j\}) = \varepsilon$ WHERE ε IS THE CONSTANT VALUE OF f

$n \to n+1$ SAME PROOF GIVEN F

$F_a : [[a+1, \omega)]^n \longrightarrow k$

$F_a(X) := F(\{a\} \cup X)$

F_{x_k} IS CONSTANT ON $[X_k]^n$ WITH VALUE $f(k)$

f CONSTANT ON I WITH VALUE ε

$H = \{x_i : i \in I\}$

$X \in [I]^{n+1}$, $i = \min X$:

$F(\{x_j : j \in X\}) = F_{x_i}(\{x_j : j \in X \setminus \{i\}\}) = f(i) = \varepsilon$

WE CALL H HOMOGENEOUS FOR F.

NOTATION

$$\kappa \longrightarrow (\lambda)^n_m$$

FOR EVERY $F : [\kappa]^n \longrightarrow m$

THERE IS $H \subseteq \kappa$ WITH $|H| = \lambda$ AND

$F \restriction [H]^n$ IS CONSTANT

SO $6 \longrightarrow (3)^2_2$ $\aleph_0 \longrightarrow (\aleph_0)^n_k$

FOR ALL $n, k \in \omega$.