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CARDINAL EXPONENTIATION

- κ REGULAR AND $\lambda < \kappa$

  THEN $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ (SETS)

  (EACH $f: \lambda \to \kappa$ IS BOUNDED)

  SO $\kappa^\lambda = \Sigma_{\alpha<\kappa} |\alpha|^\lambda$ (CARDINALS)

  κ SUCCESSOR: $\Sigma_{\alpha<\kappa} |\alpha|^\lambda = \kappa \cdot \sup_{\alpha<\kappa} |\alpha|^\lambda$
  $\kappa = \mu^+$ $\qquad = \kappa \cdot \mu^\lambda$

  OR $(\mu^+)^\lambda = \mu^+ \cdot \mu^\lambda$ IF $\lambda \leq \mu$

  OR $\beth_{\alpha+1}^{\beth_\beta} = \beth_{\alpha+1} \cdot \beth_\alpha^{\beth_\beta}$ IF $\beta \leq \alpha$

  ALSO IF $\beta > \alpha$: BOTH SIDES EQUAL $2^{\beth_\beta}$.

5.19 κ A LIMIT CARDINAL $\lambda \geq \mathrm{cf}\kappa$:

$$\kappa^\lambda = (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\mathrm{cf}\kappa}$$

□ $\kappa = \Sigma_{i<\mathrm{cf}\kappa} \kappa_i$ ($\kappa_i < \kappa$ ALWAYS)

$$\kappa^\lambda \leq (\Pi_{i<\mathrm{cf}\kappa} \kappa_i)^\lambda = \Pi_{i<\mathrm{cf}\kappa} \kappa_i^\lambda$$
$$\leq \Pi_{i<\mathrm{cf}\kappa} (\lim_{\alpha\to\kappa} \alpha^\lambda)$$
$$= (\lim_{\alpha\to\kappa} \alpha^\lambda)^{\mathrm{cf}\kappa} \leq (\kappa^\lambda)^{\mathrm{cf}\kappa} = \kappa^\lambda \text{ ok.}$$

5.20: λ INFINITE

- $\kappa \leq \lambda \Rightarrow \kappa^\lambda = 2^\lambda$ (WE KNOW THAT)
- IF $\kappa \leq \mu^\lambda$ FOR SOME $\mu < \kappa$ THEN $\kappa^\lambda = \mu^\lambda$
- IF $\kappa > \lambda$ AND $\mu^\lambda < \kappa$ FOR ALL $\mu < \kappa$

  THEN:

  $\lambda < \mathrm{cf}\kappa \to \kappa^\lambda = \kappa$

  $\mathrm{cf}\kappa \leq \lambda \to \kappa^\lambda = \kappa^{\mathrm{cf}\kappa}$

□
-
- $\mu^\lambda \leq \kappa^\lambda \leq \mu^{\lambda\cdot\lambda} = \mu^\lambda$
- $\kappa = \mu^+$: $\kappa^\lambda = \kappa \cdot \mu^\lambda = \kappa$.

  κ LIMIT $\quad \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$

  $\mathrm{cf}\kappa > \lambda$: $\kappa^\lambda = \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$

  $\mathrm{cf}\kappa \leq \lambda$: USE 5.19 : $\kappa^{\mathrm{cf}\kappa}$

5.21: κ^λ IS 2^λ, OR κ, OR $\beth(\mu)$ FOR SOME μ WITH $\mathrm{cf}\mu \leq \lambda < \mu$.

WE HAVE

SUCCESSORS

LIMITS

STRONG LIMITS: $\lambda < \kappa \rightarrow 2^\lambda < \kappa$

or $\lambda, \nu < \kappa \rightarrow \lambda^\nu < \kappa$

AND THEN $2^\kappa = \kappa^{\mathrm{cf}\,\kappa}$ $\left(2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa}\right)$

WEAKLY INACCESSIBLE: REGULAR + LIMIT

(STRONGLY) INACCESSIBLE: REGULAR + STRONG LIMIT

SINGULAR CARDINAL HYPOTHESIS

SCH: κ SINGULAR AND $2^{\mathrm{cf}\,\kappa} < \kappa$

IMPLIES $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$

(FOLLOWS FROM GCH)

5.22 ASSUME SCH.

- κ SINGULAR: $2^\kappa = 2^{<\kappa}$ (EVENTUALLY CONSTANT)

  $2^\kappa = (2^{<\kappa})^+$ (OTHERWISE

  APPLY 5.19 $\lambda = 2^{<\kappa}$

  so $2^\kappa = \lambda$ or $2^\kappa = \lambda^{\mathrm{cf}\,\kappa}$ AND $\mathrm{cf}\,\lambda = \mathrm{cf}\,\kappa$

  so $\lambda^{\mathrm{cf}\,\lambda} = \lambda^+$

- κ, λ INFINITE
  - $\kappa \leq 2^\lambda \rightarrow \kappa^\lambda = 2^\lambda$ (KNOWN)
  - $2^\lambda < \kappa \wedge \lambda < \mathrm{cf}\,\kappa \rightarrow \kappa^\lambda = \kappa$
  - $2^\lambda < \kappa \wedge \lambda \geq \mathrm{cf}\,\kappa \rightarrow \kappa^\lambda = \kappa^+$

  $\kappa = \nu^+$: $\nu^\lambda \leq \kappa$; $\nu^\lambda \in \{2^\lambda, \nu, \nu^+\}$

  $\kappa^\lambda = \kappa \cdot \nu^\lambda = \kappa$

  κ LIMIT $\nu^\lambda < \kappa$ ALL $\nu < \kappa$

  5.20 $\lambda < \mathrm{cf}\,\kappa \rightarrow \kappa^\lambda = \kappa$

  $\lambda \geq \mathrm{cf}\,\kappa$ $\kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$

  so $2^{\mathrm{cf}\,\kappa} \leq 2^\lambda < \kappa$

  AND so $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$.

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STATIONARY SETS

USUALLY: κ IS REGULAR UNCOUNTABLE

$C \subseteq \kappa$ IS CLOSED UNBOUNDED IN κ

IF - $\sup C = \kappa$ (UNBOUNDED)

- IF $\alpha < \kappa$ AND $\alpha = \sup(C \cap \alpha)$ THEN $\alpha \in C$ (CLOSED)

$S \subseteq \kappa$ IS STATIONARY IN κ

IF $S \cap C \neq \emptyset$ FOR ALL CUB SETS C.

P.2 C AND D CUB $\rightarrow$ $C \cap D$ IS CUB

- CLOSED: IF $\alpha = \sup(C \cap D \cap \alpha)$

THEN ALSO $\alpha = \sup(C \cap \alpha)$ SO $\alpha \in C$

AND $\alpha = \sup(D \cap \alpha)$ SO $\alpha \in D$.

- UNBOUNDED: LET $\alpha < \kappa$.

LET $\alpha_0 = \min\{\gamma \in C : \gamma > \alpha\}$

$\alpha_{2n+1} = \min\{\delta \in D : \delta > \alpha_{2n}\}$

$\alpha_{2n+2} = \min\{\gamma \in C : \gamma > \alpha_{2n+1}\}$

SO $\alpha < \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots$

LET $\beta = \sup_n \alpha_n$: $\beta < \kappa$ (CF $\kappa > \omega$)

$\beta = \sup_n \alpha_{2n}$ SO $\beta \in C$

$\beta = \sup_n \alpha_{2n+1}$ SO $\beta \in D$

P.3 IF $\lambda < \kappa$ AND C_α IS CUB IN κ FOR $\alpha < \lambda$ THEN

$\bigcap_{\alpha < \lambda} C_\alpha$

IS CUB

- CLOSED: CLEAR
- UNBOUNDED: LET $\alpha < \kappa$

RECURSIVELY: $\beta_0 \in C_0$ $\beta_1 = \min\{\gamma \in C_1 : \gamma > \beta_0\}$

$\beta_{\lambda \cdot n + \alpha} = \min\{\gamma \in C_\alpha : (\forall \iota < \lambda \cdot n + \alpha)(\gamma > \beta_\iota)\}$

$\beta = \sup\{\beta_\gamma : \gamma < \lambda \cdot \omega\} = \sup\{\beta_{\lambda \cdot n + \alpha} : \alpha < \lambda\}$ (EACH α)

SO $\beta \in \bigcap_{\alpha < \lambda} C_\alpha$.

DIAGONAL INTERSECTION OF $\{X_\alpha : \alpha < \kappa\}$

$$\Delta_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : (\forall \alpha < \xi)(\xi \in X_\alpha)\}$$

8.4 $\langle C_\alpha : \alpha < \kappa \rangle$ SEQUENCE OF CLUBS

THEN $\Delta_{\alpha<\kappa} C_\alpha$ IS CLUB

□ $D_\alpha = \bigcap_{\xi \le \alpha} C_\xi$: THEN $\Delta_{\alpha<\kappa} D_\alpha = \Delta_{\alpha<\kappa} C_\alpha =: C$

- CLOSED: SAY $\alpha = \sup C \cap \alpha$

  LET $\xi < \alpha$

  IF $\beta \in C \cap \alpha$ AND $\xi < \beta$ THEN $\beta \in C_\xi$

  SO $\alpha = \sup C_\xi \cap \alpha$ AND HENCE $\alpha \in C_\xi$

  WE FIND $\alpha \in C_\xi$ FOR ALL $\xi < \alpha$

- UNBOUNDED

  LET $\delta < \kappa$

8.6 $f : S \to ORD$ IS REGRESSIVE

IF $(\forall \alpha \in S)(\alpha > 0 \to f(\alpha) < \alpha)$

8.7 FODOR - PRESSING-DOWN LEMMA

IF $S \subseteq \kappa$ IS STATIONARY AND $f : S \to \kappa$

IS REGRESSIVE THEN THERE IS

A STATIONARY SET T ON WHICH f IS CONSTANT

□ Suppose not

So FOR EVERY $\gamma < \kappa$ THERE IS A CUB C_γ

SUCH THAT $C_\gamma \cap \{\alpha \in S : f(\alpha) = \gamma\} = \emptyset$

LET $C = \Delta_{\gamma < \kappa} C_\gamma$ AND TAKE $\alpha \in S \cap C$ $(\alpha > 0)$

THEN $\alpha \in C_{f(\alpha)}$ HENCE $f(\alpha) \neq f(\alpha)$ ... □

κ REGULAR UNCOUNTABLE $\lambda < \kappa$ REGULAR

$E^\kappa_\lambda = \{\alpha < \kappa : cf\,\alpha = \lambda\}$

THIS SET IS STATIONARY.

So $E^\kappa_{\omega_0}$ AND $E^\kappa_{\omega_1}$ ARE DISJOINT STATIONARY SETS

IF $\kappa \geq \aleph_2$: THE CUB FILTER IS NOT

AN ULTRAFILTER

8.8 EVERY STATIONARY SUBSET OF E^κ_ω IS THE

UNION OF κ DISJOINT STATIONARY SUBSETS

□ LET $W \subseteq E^\kappa_\omega$ BE STATIONARY

FOR EACH $\alpha \in W$ CHOOSE $\langle a(\alpha,n) : n < \omega \rangle$

INCREASING AND COFINAL IN α.

• THERE IS AN n SUCH THAT FOR ALL $\eta < \kappa$

$W(n,\eta) = \{\alpha \in W : a(\alpha,n) \geq \eta\}$

IS STATIONARY

IF NOT LET $\eta_n < \kappa$ AND C_n BE CUB

SUCH THAT $C_n \cap W(n,\eta_n) = \emptyset$