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CARDINAL ARITHMETIC

- FOR INFINITE CARDINALS $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$
- 2^λ AND κ^λ ARE MORE INTERESTING.

5.6: IF $2 \le \kappa \le \lambda$ (EVEN IF $\kappa \le 2^\lambda$) AND $\lambda \ge \aleph_0$

THEN $\kappa^\lambda = 2^\lambda$

PROOF $2^\lambda \le \kappa^\lambda \le (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$ $\square$

EVALUATING κ^λ IS INTERESTING IF $2^\lambda < \kappa$.

ALL WE CAN SAY $\kappa \le \kappa^\lambda \le 2^\kappa$

BUT ALSO $\kappa^{\mathrm{cf}\,\kappa} > \kappa$

HENCE $\lambda \ge \mathrm{cf}\,\kappa \to \kappa^\lambda > \kappa$.

SOMETIMES IT IS EASIER TO STUDY $[\kappa]^\lambda$

$$[A]^\lambda = \{X \subseteq A : |X| = \lambda\}$$

($|A| \ge \lambda$ OF COURSE)

5.7 IF $|A| = \kappa \ge \lambda$ THEN $|[A]^\lambda| = \kappa^\lambda$

PROOF:
- $A^\lambda \subseteq [\lambda \times A]^\lambda$ AND $|\lambda \times A| = |A| = \kappa$

  SO $\kappa^\lambda \le |[\kappa]^\lambda|$
- AC FOR EVERY $X \in [A]^\lambda$ CHOOSE ONE $f_X \in A^\lambda$ WITH $X = \operatorname{ran} f_X$

  SO $|[A]^\lambda| \le |A^\lambda|$ AS $X \mapsto f_X$ IS INJECTIVE. $\square$

IF λ IS A LIMIT THEN DEFINE

$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu < \lambda, \mu \text{ CARDINAL}\}$$

IF $|A| \ge \kappa$ THEN DEFINE

$$[A]^{<\kappa} = \mathcal{P}_\kappa(A) = \{X \subseteq A : |X| < \kappa\}$$

NOTE $[A]^{<\kappa} = \bigcup\{[A]^\mu : \mu < \kappa, \mu \text{ CARDINAL}\}$

SO $|[A]^{<\kappa}| \ge \sup\{|A|^\mu : \mu < \kappa, \mu \text{ CARDINAL}\}$

$= |A|^{<\kappa}$.

WE SHALL SEE, VIA 5.8, $|[A]^{<\kappa}| = |A|^{<\kappa}$.

INFINITE SUMS:

$$\Sigma_{i\in I}\kappa_i = |\bigcup_{i\in I}\{i\}\times X_i|$$

(INDEPENDENT OF CHOICE OF SETS X_i WITH $|X_i|=\kappa_i$)

- IF $\kappa_i = \kappa$ FOR ALL $i<\lambda$ THEN

$$\Sigma_{i<\lambda}\kappa_i = \kappa\cdot\lambda$$

5.8 IF λ IS INFINITE, $\kappa_i>0$ FOR ALL $i<\lambda$ THEN

$$\Sigma_{i<\lambda}\kappa_i = \lambda\cdot\sup_{i<\lambda}\kappa_i$$

PROOF LET $\kappa = \sup_{i<\lambda}\kappa_i$

- NOTE $\bigcup_{i<\lambda}\{i\}\times X_i \subseteq \lambda\times\kappa$

  SO $\Sigma_{i<\lambda}\kappa_i \le \lambda\cdot\kappa$.

- $\lambda = \Sigma_{i<\lambda}1 \le \Sigma_{i<\lambda}\kappa_i$

  $\kappa_j \le \Sigma_{i<\lambda}\kappa_i$ FOR EACH j

  SO $\sup_{i<\lambda}\kappa_i \le \Sigma_{i<\lambda}\kappa_i$

  TOGETHER $\lambda\cdot\sup_{i<\lambda}\kappa_i \le \Sigma_{i<\lambda}\kappa_i$ $\square$

SO IF $\lambda \le \sup_{i<\lambda}\kappa_i$

THEN $\Sigma_{i<\lambda}\kappa_i = \sup_{i<\lambda}\kappa_i$

SO κ IS SINGULAR IFF $\kappa = \Sigma_{i<\lambda}\kappa_i$

WITH $\lambda<\kappa$ AND $\kappa_i<\kappa$ ALL i.

PRODUCTS:

$$\Pi_{i\in I}\kappa_i = |\Pi_{i\in I}\kappa_i|$$

THE <u>SET</u> $\Pi_{i\in I}X_i$ IS $\{f : f: I\to\bigcup_{i\in I}X_i \wedge (\forall i\in I)(f(i)\in X_i)\}$

$$\Pi_{i\in I}\kappa_i^{\lambda} = (\Pi_{i\in I}\kappa_i)^{\lambda}$$

$$\Pi_{i\in I}\kappa^{\lambda_i} = \kappa^{\Sigma_{i\in I}\lambda_i}$$

ASSOCIATIVITY / COMMUTATIVITY:

IF $I = \bigcup_{j \in J} A_j$ (DISJOINT)

THEN $\prod_{i \in I} \kappa_i = \prod_{j \in J} \left(\prod_{i \in A_j} \kappa_i \right)$

- $\kappa_i \geq 2$ FOR ALL i:

$$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa_i$$

- DEFINE $f : \bigcup_{i \in I} \{i\} \times \kappa_i \to \prod_{i \in I} \kappa_i$

BY $f(i, \alpha)(j) = \begin{cases} 0 & j \neq i \\ \alpha & j = i \end{cases} \quad \alpha > 0$

$$f(i, 0)(j) = \begin{cases} 1 & j \neq i \\ 0 & j = i \end{cases}$$

VERIFY THAT f IS INJECTIVE

5.9: IF λ IS INFINITE AND $\langle \kappa_i : i < \lambda \rangle$ IS NONDECREASING WITH $\kappa_0 > 0$ THEN

$$\prod_{i < \lambda} \kappa_i = (\sup \kappa_i)^\lambda$$

PROOF PUT $\kappa = \sup_{i < \lambda} \kappa_i$ SO $\prod_{i < \lambda} \kappa_i \leq \kappa^\lambda$.

FOR $\geq$, PARTITION λ INTO λ SETS OF CARDINALITY λ (VIA A BIJECTION $G : \lambda \to \lambda \times \lambda$)

SAY $\lambda = \bigcup_{j < \lambda} A_j$

THEN $\kappa = \sup_{i \in A_j} \kappa_i$ FOR ALL j

AND

$$\begin{aligned} \prod_{i < \lambda} \kappa_i &= \prod_{j < \lambda} \left(\prod_{i \in A_j} \kappa_i \right) \\ &\geq \prod_{j < \lambda} \left(\sup_{i \in A_j} \kappa_i \right) \\ &= \kappa^\lambda \end{aligned}$$

BECAUSE $\prod_{i \in A_j} \kappa_i \geq \kappa_{i_0}$ FOR EACH i_0

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KÖNIG'S INEQUALITY (5.10)

ASSUME $\kappa_i < \lambda_i$ FOR ALL $i \in I$

THEN $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$

PROOF

- WLOG $\kappa_i > 0$ FOR ALL i
- SO $\lambda_i \geq 2$ FOR ALL i

NOW USE THE SAME INJECTION AS A FEW MINUTES AGO: $f : \bigcup_{i \in I} \{i\} \times \kappa_i \to \prod_{i \in I}$

$$f(i,0)(j) = \begin{cases} 1 & j \neq i \\ 0 & j = i \end{cases}$$

$$\alpha > 0 : f(i,\alpha)(j) = \begin{cases} 0 & j \neq i \\ \alpha & j = i \end{cases}$$

SO $\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$

- LET $f : \bigcup_{i \in I} \kappa_i \longrightarrow \prod_{i \in I} \lambda_i$ BE ARBITRARY

FOR EACH i CONSIDER

$$\{ f(i,\alpha)(i) : \alpha < \kappa_i \} \subseteq \lambda_i$$

THIS SET HAS CARDINALITY κ_i OR LESS

HENCE CARDINALITY LESS THAN λ_i

SO $\alpha_i = \min \lambda_i \setminus \{ f(i,\alpha)(i) : \alpha < \kappa_i \}$

CONSEQUENCES

5.11 $\kappa < 2^\kappa$: $\kappa_i = 1$, $\lambda_i = 2$ $i < \kappa$

5.12 $\mathrm{cf}\, 2^\kappa > \kappa$: $\kappa_i < \lambda_i = 2^\kappa$ FOR $i < \kappa$.

THEN $\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \lambda_i = 2^{\kappa \cdot \kappa} = 2^\kappa$.

5.13 $\mathrm{cf}\,(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$ SAME PROOF

5.14 $\kappa^{\mathrm{cf}\,\kappa} > \kappa$: $\kappa_i < \kappa$ FOR $i < \mathrm{cf}\,\kappa$

WITH $\sum_{i<\mathrm{cf}\,\kappa} \kappa_i = \kappa$

THEN $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i < \prod_{i<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}$

FOR EXAMPLE $\aleph_\omega^{\aleph_0} > \aleph_\omega$

THE CONTINUUM FUNCTION

WE KNOW $2^{\aleph_\alpha} > \aleph_\alpha$ OR $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$

GCH SAYS $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ FOR <u>ALL</u> α.

IN THAT CASE

5.15 ASSUME GCH AND LET $\kappa, \lambda \geq \aleph_0$

(i) IF $\kappa \leq \lambda$ THEN $\kappa^\lambda = \lambda^+$

(ii) IF $\mathrm{cf}\,\kappa \leq \lambda < \kappa$ THEN $\kappa^\lambda = \kappa^+$

(iii) IF $\lambda < \mathrm{cf}\,\kappa$ THEN $\kappa^\lambda = \kappa$.

PROOF (i) : $\kappa^\lambda = 2^\lambda$

(ii) $\kappa < \kappa^\lambda \leq 2^\kappa = \kappa^+$

(iii) $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

$$\begin{aligned} \text{so } |\kappa^\lambda| &= \kappa \cdot \sup\{|\alpha|^\lambda : \alpha < \kappa\} \\ &\leq \kappa \cdot \sup\{2^{|\alpha| \cdot \lambda} : \alpha < \kappa\} \\ &\leq \kappa \cdot \sup\{(|\alpha| \cdot \lambda)^+ : \alpha < \kappa \\ &= \kappa \cdot \kappa \\ &= \kappa. \end{aligned}$$

NOW IN GENERAL

5.16 (i) IF $\kappa < \lambda$ THEN $2^\kappa \leq 2^\lambda$

(ii) CF $2^\kappa > \kappa$

(iii) IF κ IS A LIMIT THEN $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\kappa}$

PROOF (i) CLEAR (CANNOT BE IMPROVED TO $2^\kappa < 2^\lambda$)

(ii) 5.12

(iii) SAY $\kappa = \Sigma_{i<\mathrm{cf}\kappa} \kappa_i$ ($\kappa_i < \kappa$ ALWAYS)

$$2^\kappa = 2^{\Sigma_i \kappa_i} = \Pi_i 2^{\kappa_i} \leq \Pi_i 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}\kappa}$$
$$\leq (2^\kappa)^{\mathrm{cf}\kappa} = 2^\kappa. \square$$

5.17 IF κ IS SINGULAR AND THERE ARE $\mu < \kappa$ AND λ SUCH THAT

$$(\forall \nu < \kappa)(\mu \leq \nu \rightarrow 2^\nu = \lambda)$$

THEN $2^\kappa = \lambda$.

PICK $\nu \geq \mathrm{cf}\kappa$ WITH $2^\nu = \lambda = 2^{<\kappa}$

THEN $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\kappa} \leq (2^\nu)^\nu = 2^\nu = \lambda$ $\square$

$\lambda(\kappa) = \kappa^{\mathrm{cf}\kappa}$

IF 2^μ IS NOT EVENTUALLY CONSTANT BELOW κ

THEN $\lambda = 2^{<\kappa} = \lim_{\mu \to \kappa} 2^\mu$ κ LIMIT

AND $\mathrm{cf}\lambda = \mathrm{cf}\kappa$

AND SO $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\kappa} = \lambda^{\mathrm{cf}\lambda}$

IF κ IS REGULAR THEN $\kappa = \mathrm{cf}\kappa$

SO $2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cf}\kappa}$

5.18 (i) κ SUCCESSOR: $2^\kappa = \lambda(\kappa)$ (REGULAR)

(ii) κ LIMIT $\wedge$ 2^μ EV. CONSTANT BELOW κ:

$$2^\kappa = 2^{<\kappa} \cdot \lambda(\kappa)$$

(iii) κ LIMIT $\wedge$ 2^μ NOT EV. CONSTANT BELOW κ:

$$2^\kappa = \lambda(2^{<\kappa}).$$