HOMEWORK 5

SET THEORY

► **1** (Jech: 2.13). An ordinal γ is called a *gamma number* (also known as *additively indecomposable ordinal* or *principal number of addition*) if for all $\alpha, \beta < \gamma$, we have $\alpha + \beta < \gamma$. Show that for $\gamma > 0$, the following are equivalent:

(i) γ is a gamma number,
(ii) for all $\alpha < \gamma$, we have $\alpha + \gamma = \gamma$,
(iii) there is some ξ such that $\gamma = \omega^{\xi}$.

► **2**. Let α and β be ordinals and let $\alpha = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$ and $\beta = \omega^{\beta_1} \cdot \ell_1 + \cdots + \omega^{\beta_m} \cdot \ell_m$ be the Cantor normal forms of α and β, respectively. Merge the two sequences to a finite descending sequence of ordinals $\gamma_1 > \cdots > \gamma_k$ such that $\{\gamma_i : 1 \leq i \leq k\} = \{\alpha_i : 1 \leq i \leq n\} \cup \{\beta_i : 1 \leq i \leq m\}$,

$$\alpha = \omega^{\gamma_1} \cdot p_1 + \cdots + \omega^{\gamma_k} \cdot p_k \text{ and}$$
$$\beta = \omega^{\gamma_1} \cdot q_1 + \cdots + \omega^{\gamma_k} \cdot q_k,$$

where

$$p_i := \begin{cases} k_j & \text{if } \gamma_i = \alpha_j, \\ 0 & \text{if } \gamma_i \notin \{\alpha_1, \ldots, \alpha_n\}, \end{cases} \quad q_i := \begin{cases} \ell_j & \text{if } \gamma_i = \beta_j, \text{ and} \\ 0 & \text{if } \gamma_i \notin \{\beta_1, \ldots, \beta_m\}. \end{cases}$$

Define

$$\alpha \mathbin{\#} \beta := \omega^{\gamma_1} \cdot (p_1 + q_1) + \cdots + \omega^{\gamma_k} \cdot (p_k + q_k).$$

The operation is called the *Hessenberg addition* or *natural addition.* Show that the operation # is commutative and that $\alpha \mathbin{\#} \beta \geq \max(\alpha + \beta, \beta + \alpha)$. Find examples for α and β such that

(a) $\alpha \mathbin{\#} \beta = \alpha + \beta \neq \beta + \alpha$,
(b) $\alpha \mathbin{\#} \beta > \max(\alpha + \beta, \beta + \alpha)$.

► **3**. We define the following relation for ordinal numbers α, β, α' and β':

$$\begin{aligned}(\alpha, \beta) <^* (\alpha', \beta') :\iff & \max(\alpha, \beta) < \max(\alpha', \beta') \text{ or} \\ & (\max(\alpha, \beta) = \max(\alpha', \beta') \text{ and } \alpha < \alpha') \text{ or} \\ & (\max(\alpha, \beta) = \max(\alpha', \beta') \text{ and } \alpha = \alpha' \text{ and } \beta < \beta').\end{aligned}$$

(a) Show that for every ordinal ξ the structure $(\xi \times \xi, <^*)$ is a well-order.

We denote by $g(\xi)$ the unique ordinal isomorphic to $(\xi \times \xi, <^*)$ and by $G_\xi : \xi \times \xi \to g(\xi)$ the unique isomorphism between ξ and $g(\xi)$. Show that for every $\xi < \eta$ and $\alpha, \beta \in \xi$ we have

$$G_\xi(\alpha, \beta) = G_\eta(\alpha, \beta).$$

Therefore we can define the following class function:

$$G : (\alpha, \beta) \mapsto G_{\max(\alpha,\beta)+1}(\alpha, \beta).$$

(b) Show that the class function G maps pairs of ordinals to ordinals and satisfies the conditions for (i) injectivity, (ii) order preservation, and (iii) surjectivity, i.e., (i) for all ordinals $\alpha, \beta, \alpha', \beta'$, if $G((\alpha, \beta)) = G((\alpha', \beta'))$, then $(\alpha, \beta) = (\alpha', \beta')$; (ii) for all ordinals $\alpha, \beta, \alpha', \beta'$, we have $G((\alpha, \beta)) < G((\alpha', \beta'))$ if and only if $(\alpha, \beta) <^* (\alpha', \beta')$; and (iii) every ordinal is $G((\alpha, \beta))$ for some α and β.
(c) Show by induction that for every infinite cardinal κ we have $g(\kappa) = \kappa$. [*Hint.* First show that if the claim holds for infinite *cardinals* $\lambda < \kappa$, then for every infinite *ordinal* $\alpha < \kappa$, we have $|\alpha \times \alpha| = |\alpha|$.]
(d) Finally, prove Hessenberg's Theorem: for every infinite cardinal κ, we have $|\kappa \times \kappa| = \kappa$.

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