HOMEWORK 14 SOLUTION NOTES

SET THEORY

Disclaimer: The following are rough solution notes for homework 14, not template solutions.

Let us start by remembering some definitions and facts: For each infinite κ, we defined:

$$\mathbf{H}_\kappa = \{x \mid |\mathrm{tcl}(\{x\})| < \kappa\}.$$

First note that $\mathrm{tcl}(x) = x \cup \bigcup\{\mathrm{tcl}(y) \mid y \in x\}$. So, while in general $\mathrm{tcl}(x) \neq \mathrm{tcl}(\{x\})$, we get for all κ and all x that

$$|\mathrm{tcl}(\{x\})| < \kappa \iff |\mathrm{tcl}(x)| < \kappa,$$

so, in particular, $\mathbf{H}_\kappa = \{x \mid |\mathrm{tcl}(x)| < \kappa\}$. Furthermore, in class, we defined the following notion for an infinite cardinal κ: a set x is of *hereditary cardinality* $<\kappa$ iff $|x| < \kappa$ and $\forall y \in x |\mathrm{tcl}(y)| < \kappa$.

Fact 1. If κ is regular then $x \in \mathbf{H}_\kappa$ iff x is of hereditary cardinality $<\kappa$.

Note that if κ is singular with $\mathrm{cf}(\kappa) = \lambda < \kappa$, then there is a sequence in $(\alpha_\beta)_{\beta\in\lambda}$ of ordinals in κ such that $\kappa = \sup\{\alpha_\beta \mid \beta \in \lambda\}$. But then $X := \{\alpha_\beta \mid \beta \in \lambda\} \notin \mathbf{H}_\kappa$ since $\mathrm{tcl}(X) = \kappa$ but the set X is trivially of hereditary cardinality $<\kappa$. So regularity is actually needed in Fact 1.

▶ **1**. Remember the *beth function* defined by transfinite recursion as follows:

$$\begin{aligned} \beth_0 &= \omega, \\ \beth_{\alpha+1} &= 2^{\beth_\alpha}, \\ \beth_\delta &= \bigcup_{\alpha<\delta} \beth_\alpha \text{ for limit ordinals } \delta. \end{aligned}$$

A cardinal λ is called a *beth fixed point* if $\beth_\lambda = \lambda$.

(1) Show that a cardinal κ is a strong limit if and only if there is a limit ordinal δ such that $\beth_\delta = \kappa$.
(2) Let κ be a regular cardinal. Show that there is a beth fixed point λ such that $\mathrm{cf}(\lambda) = \kappa$.

Proof. (1) Assume that $\kappa = \beth_\delta$ for some limit δ. Let $\alpha < \kappa$. Note that by definition $\kappa = \bigcup_{\beta<\delta} \beth_\beta$. Therefore there is $\beta < \delta$ such that $\alpha < \beth_\beta$. Then $2^\alpha \leq 2^{\beth_\beta} = \beth_{\beta+1} < \beth_\delta = \kappa$.

On the other hand if κ is strong limit, let $A = \{\gamma' \mid \beth_{\gamma'} \geq \kappa\}$. Note that $\kappa + 1 \in A$ so $A \neq \varnothing$. Let δ be the minimum of A. First note that δ is limit. Suppose not then $\delta = \alpha + 1$ and $\beth_\delta = \beth_{\alpha+1} \geq \kappa$. But then $\beth_\alpha < \kappa$ and $2^{\beth_\alpha} = \beth_\delta \geq \kappa$ which contradicts the fact that κ was strong limit. Moreover $\beth_\delta = \kappa$. If $\kappa < \beth_\delta$ then exists $\beta < \delta$ such that $\kappa < \beth_\beta$ but this contradicts the minimality of δ.

(2) Let κ be regular. Let $(\alpha_\beta)_{\beta\in\kappa}$ be a strictly increasing sequence of κ many fixed points of $\beth$ (they exist because $\beth$ is normal). Let $\lambda = \sup\{\alpha_\beta \mid \beta < \kappa\}$. First we show that $\lambda = \beth_\lambda$. By the continuity of $\beth$ and the definition of λ we have: $\lambda = \sup\{\alpha_\beta \mid \beta < \kappa\} = \sup\{\beth_{\alpha_\beta} \mid \beta < \kappa\} = \beth_{\sup\{\alpha_\beta \mid \beta<\kappa\}} = \beth_\lambda$. Moreover, since κ is regular any sequence of length $<\kappa$ in λ is bounded by α_β for some $\beta < \kappa$ so λ has cofinality κ as desired. □

▶ **2**. A formula Φ is called *serial* if for all x there is a y such that $\Phi(x, y)$. If Φ is a serial formula, the following formula is called the *Axiom of Collection for* Φ:

$$\forall X \exists Y \forall x \in X \exists y \in Y \varphi(x, y).$$

If the Axiom of Collection for Φ holds, we say that Y *collects* X *with respect to* Φ. If M is a set, we say that M is *closed under Collection* if for every serial formula Φ and every $X \in M$, then there is a $Y \in M$ that collects X with respect to Φ.

(1) Show that for every serial formula Φ, the other axioms of set theory imply the Axiom of Collection for Φ. (Mention explicitly in the proof which axioms you used.)
(2) Show that for each *regular* cardinal κ, $\mathbf{H}_\kappa$ is closed under Collection. (*Note that the homework set was missing the word "regular".*)

Proof. (1) We work in ZF. For each x, let C_x be the class $\{y \mid \varphi(x,y)\}$. We define the following class $C_x^* := \{y \mid y \in C_x \wedge \forall z \in C_x \varrho(z) \leq \varrho(y)\}$. Note that for all x the class $C_x^* \subset \mathbf{V}_\beta$ where $\beta = \inf\{\varrho(z) \mid z \in C_x\}$. Therefore C_x^* is a set. Now let X be a set. Define by replacement $\{C_x^* \mid x \in X\}$, and let by union $Y := \bigcup_{x\in X} C_x$. Then trivially for every $x \in X$ there is $y \in C_x^* \subset Y$ such that $\varphi(x,y)$ as desired.

(2) As mentioned in the exercise class we need to assume κ regular. In this case we have that $\mathbf{H}_\kappa$ is closed under all the axioms but Power set.

Following the previous proof for each $x \in X \in \mathbf{H}_\kappa$ define $Y_x := C_x^* \cap \mathbf{H}_\kappa$. Note that this is not empty set by assumption. By Replacement, let $Y := \{Y_x \mid x \in X\}$. Then, since $\mathbf{H}_\kappa$ is closed under Replacement, we have $Y \in \mathbf{H}_\kappa$. Finally, let $Y' := \bigcup_{x\in X} Y_x$. Then, since $\mathbf{H}_\kappa$ is closed under Union, we get $Y' \in \mathbf{H}_\kappa$. As for part 1 is not hard to see that Y' collects X as desired. □

▶ **3**. Next week we will prove that if κ is an inaccessible cardinal, then $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. Check whether the two converses hold or not, i.e.,

(1) "if $\mathbf{V}_\kappa = \mathbf{H}_\kappa$, then κ is regular" and
(2) "if $\mathbf{V}_\kappa = \mathbf{H}_\kappa$, then κ is a strong limit cardinal".

For each of the statements, either give a proof or a counterexample.

Proof. (1) The answer is "No". First note that for every α we have $|\mathbf{V}_{\omega+\alpha}| = \beth_{\omega+\alpha}$ (induction on α). We will prove that if $\kappa = \beth_\kappa$ then $\mathbf{V}_\kappa = \mathbf{H}_\kappa$ (for $\kappa > \omega$, the converse can also be proved).

First we prove that for every infinite cardinal κ we have $\mathbf{H}_\kappa \subseteq \mathbf{V}_\kappa$. In class we proved that this is true for regular κ. Suppose now that κ is singular, i.e., in particular a limit cardinal. We have $X \in \mathbf{H}_\kappa$ iff $|\mathrm{tcl}(X)| < \kappa$ iff there is $\lambda < \kappa$ and $|\mathrm{tcl}(X)| < \lambda^+$. But since λ^+ is regular we have $X \in \mathbf{V}_{\lambda^+} \subseteq \mathbf{V}_\kappa$.

For the other direction, let $X \in \mathbf{V}_\kappa$. Then $X \in \mathbf{V}_\alpha$ for some $\alpha < \kappa$. But then $\mathrm{tcl}(X) \subseteq \mathbf{V}_\alpha$ ($X \subseteq \mathbf{V}_\alpha$ and $\mathbf{V}_\alpha$ transitive) so $|\mathrm{tcl}(X)| \leq |\mathbf{V}_\alpha| = \beth_{\omega+\alpha} < \beth_\kappa = \kappa$. Then $X \in \mathbf{H}_\kappa$.

Now take any singular beth fixed point $\kappa = \beth_\kappa$. By the above argument, we get that $\mathbf{H}_\kappa = \mathbf{V}_\kappa$.

(2) The answer is "Yes". Assume that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. Then as showed in class, $\mathbf{H}_\kappa$ is closed under taking power set because $\mathbf{V}_\kappa$ is. Therefore if $\alpha < \kappa$ since $\alpha \in \mathbf{V}_\kappa = \mathbf{H}_\kappa$ we have $2^\alpha < \kappa$ as desired. □

▶ **4**. Let κ be a strong limit cardinal such that $\kappa \to (\kappa)^2_2$. Then κ is inaccessible. (*Note that this problem was wrongly stated on the homework set, using a weakening of $\kappa \to (\kappa)^2_2$ which is not enough to prove this result.*)

Proof. We only need to show that κ is regular. Assume $\lambda < \kappa$ and $\kappa = \bigcup_{\alpha<\lambda} B_\alpha$ with B_α pairwise disjoint and $|B_\alpha| < \kappa$. Define the following colouring $f : [\kappa]^2 \to 2$:

$$f(\alpha,\beta) = \begin{cases} 1 & \text{if } \exists\gamma < \lambda \text{ such that } \alpha,\beta \in B_\gamma; \\ 0 & \text{otherwise.} \end{cases}$$

By assumption, f has an homogeneous set H of size κ. Note that H must be of colour 1, since $\lambda < \kappa$ and we only have λ many B_αs. Finally, note that because H is homogeneous of colour 1, there must be a $\gamma < \lambda$ such that $H \subseteq B_\gamma$. But this contradicts our hypothesis. □