HOMEWORK 14

SET THEORY

► **1**. Remember the *beth function* defined by transfinite recursion as follows:

$$\beth_0 = \omega,$$
$$\beth_{\alpha+1} = 2^{\beth_\alpha},$$
$$\beth_\delta = \bigcup_{\alpha<\delta} \beth_\alpha \text{ for limit ordinals } \delta.$$

A cardinal λ is called a *beth fixed point* if $\beth_\lambda = \lambda$.

(1) Show that a cardinal κ is a strong limit if and only if there is a limit ordinal δ such that $\beth_\delta = \kappa$.
(2) Let κ be a regular cardinal. Show that there is a beth fixed point λ such that $\mathrm{cf}(\lambda) = \kappa$.

► **2**. A formula Φ is called *serial* if for all x there is a y such that $\Phi(x, y)$. If Φ is a serial formula, the following formula is called the *Axiom of Collection for* Φ:

$$\forall X \exists Y \forall x \in X \exists y \in Y \varphi(x, y).$$

If the Axiom of Collection for Φ holds, we say that Y *collects* X *with respect to* Φ. If M is a set, we say that M is *closed under Collection* if for every serial formula Φ and every $X \in M$, then there is a $Y \in M$ that collects X with respect to Φ.

(1) Show that for every serial formula Φ, the other axioms of set theory imply the Axiom of Collection for Φ. (Mention explicitly in the proof which axioms you used.)
(2) Show that for each infinite cardinal κ, $\mathbf{H}_\kappa$ is closed under Collection.

► **3**. Next week we will prove that if κ is an inaccessible cardinal, then $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. Check whether the two converses hold or not, i.e.,

(1) "if $\mathbf{V}_\kappa = \mathbf{H}_\kappa$, then κ is regular" and
(2) "if $\mathbf{V}_\kappa = \mathbf{H}_\kappa$, then κ is a strong limit cardinal".

For each of the statements, either give a proof or a counterexample.

► **4**. Let κ be a strong limit cardinal such that for all $\lambda < \kappa$, the partition relation $\kappa \to (\lambda)^2_2$ holds. Show that κ is inaccessible.

Additional questions; not part of the homework:

► **5**. Is it necessary to assume "strong limit" in Question **4** to prove inaccessibility?

► **6**. In class, we showed that for limit ordinals λ, $\mathbf{V}_\lambda$ is closed under Separation by showing that for each $x, p_1, ..., p_n \in \mathbf{V}_\lambda$ and each formula φ, the set $S_\varphi(x, p_1, ..., p_n) \in \mathbf{V}_\lambda$. We mentioned that it is not obvious that this implies that the Axiom of Separation holds in $\mathbf{V}_\lambda$ because $S_\varphi(x, p_1, ..., p_n)$ may not be the result of separating by φ from x *inside* $\mathbf{V}_\lambda$. Show that in spite of this issue, the proved closure property of $\mathbf{V}_\lambda$ is sufficient to prove that the Axiom of Separation holds in $\mathbf{V}_\lambda$.