HOMEWORK 13

SET THEORY

► **1** (Jech: 6.4). Suppose that $\varrho(x) = \alpha$ and $\varrho(y) = \beta$. Calculate the ranks of $\{x, y\}$, $\langle x, y\rangle$, $\bigcup x$, and $\mathcal{P}(x)$ in terms of α and β.

► **2**. We define:

$$\varphi_{\varnothing}(z) :\iff \forall y(y \notin z)$$

and we write $\mathrm{ind}(x) := \exists z \varphi_{\varnothing}(z) \wedge z \in x \wedge \forall w(w \in x \to \exists v(v \in x \wedge \forall y(y \in v \leftrightarrow y \in w \vee y = w)))$ for "x is an inductive set". The infinity axiom $\mathsf{Infinity}$ is then $\exists x(\mathrm{ind}(x))$. Show that $\mathbf{V}_\omega \models \neg\mathsf{Infinity}$.

Hint. Find a property Φ, prove that all elements of $\mathbf{V}_\omega$ have the property Φ, and that no inductive set has property Φ.

► **3**. Let M and N be classes such that $M \subseteq N$ and let E be a binary relation on N. We define a relation E' by $E'(x, y) \leftrightarrow M(x) \wedge M(y) \wedge E(x, y)$. The relation E' is the restriction of E to M. Let $\Phi(x_1, ..., x_n, y)$ be any formula in $n + 1$ free variables. We call Φ (M, N, E)-*absolute* if for all $x_1, ..., x_n, y \in M$, we have that

$$(M, E') \models \Phi(x_1, ..., x_n, y) \iff (N, E) \models \Phi(x_1, ..., x_n, y).$$

We define

$$\Phi_{\text{pair}}(x_1, x_2, y) :\iff \forall z(E(z, y) \leftrightarrow (x_1 = z \vee x_2 = z)) \text{ and}$$
$$\Phi_{\text{union}}(x, u) :\iff \forall z(E(z, u) \leftrightarrow \exists y(E(y, x) \wedge E(z, y))).$$

Give examples of finite sets M, N, and $E \subseteq N \times N$ such that Φ_{pair} and Φ_{union} are **not** (M, N, E)-absolute.

► **4**. Let M, N, and E be as in **Question 3**. We call M *transitive in* N if for all $x, y \in N$, if $x \in M$ and $E(y, x)$, then $y \in M$. We define:

$$\Phi_{\text{subset}}(x, y) :\iff \forall z(E(z, x) \to E(z, y)).$$

(i) Show that if M is transitive in N, then Φ_{pair}, Φ_{union} and Φ_{subset} are (M, N, E)-absolute.

(ii) Suppose

$$(N, \in) \models \mathsf{Extensionality} + \mathsf{Pairing} + \mathsf{Union} + \mathsf{Power\ set}$$

and that M is transitive in N. For every $x, y \in N$ we write $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ for the pair, the union and the power set uniquely defined in N by the axioms.

Show that if for all $x, y \in M$ we have that $\{x, y\}, \bigcup x, \mathcal{P}(x) \in M$ then

$$(M, \in') \models \mathsf{Extensionality} + \mathsf{Pairing} + \mathsf{Union} + \mathsf{Power\ set}.$$

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