HOMEWORK 7

November 1, 2017

In this text you will find solutions to the exercises of the 7th homework of Set Theory.

Exercise 1. Show that there are at least $\mathfrak{c}$ many countable order-types of linearly ordered sets.

Proof. Consider the set ${}^{\mathbb{N}}\{1,2\}$ of functions from $\mathbb{N}$ to $\{1,2\}$. Note that $|{}^{\mathbb{N}}\{1,2\}| = |{}^{\mathbb{N}}2| = 2^{\aleph_0}$ (e.g., using the map $f(0) = 1$, $f(1) = 2$). For each function $f \in {}^{\leq\mathbb{N}}\{1,2\}$ we define a countable linear order $\tau_f = (A_f, <_f)$ as follows:

$$A_f = \{\langle 2n, m\rangle \mid m \in f(n),\ n \in \mathrm{dom}(f)\} \cup \{\langle\langle 2n+1, z\rangle \mid z \in \mathbb{Z},\ n \in \mathrm{dom}(f)\},$$

and $<_f$ is the lexicographic order. Note that τ_f is trivially a linear order.

We will prove that A_f is countable. Note that the mapping $g(z) := 2n$ if $z = n$ and $g(z) := 2n+1$ if $z = -n$ shows that $\mathbb{Z}$ is countable. Now $A_f \subset \mathbb{N} \times \mathbb{Z}$ so by the Hessenberg theorem we have $|A_f| \leq |\mathbb{N} \times \mathbb{Z}| = \aleph_0$.

Now let $f \in {}^{\mathbb{N}}\{1,2\}$ and consider τ_f: in this linear order we can identify special points: every point of the form $\langle 2n, 0\rangle$ has no direct predecessor, that is, for every $x <_f \langle 2n, 0\rangle$ there is $y <_f \langle 2n, 0\rangle$ such that $x <_f y$. If $n = 0$ this is vacuously true; if $n > 0$ and $\langle i, j\rangle <_f \langle 2n, 0\rangle$ then $i < 2n$, so that $\langle i, j\rangle \leq_f \langle 2n-1, k\rangle$ for some k, but then $\langle i, j\rangle <_f \langle 2n-1, k+1\rangle <_f \langle 2n, 0\rangle$. Every other point does have direct predecessor: $\langle 2n, 1\rangle$ has $\langle 2n, 0\rangle$ as its direct predecessor and $\langle 2n+1, k\rangle$ has $\langle 2n+1, k-1\rangle$ as its direct predecessor.

It follows that if $\psi : A_f \to A_g$ defines an isomorphism we must have $\psi(2n, 0) = \langle 2n, 0\rangle$ for all n.

Next we observe that $\langle 2n, 0\rangle$ has a direct successor in τ_f iff $f(n) = 2$, and that successor is $\langle 2n, 1\rangle$.

It follows that if $\psi : A_f \to A_g$ defines an isomorphism we must have: if $f(n) = 2$ then $\psi(2n, 1) = \langle 2n, 1\rangle$ and hence $g(n) = 2$.

We conclude: if τ_f and τ_g are isomorphic then $f = g$. □

Alternative proof. Let $(O, <)$ be a linear order. Let A be a subset of O. We will say that A is a *cut* of $(O, <)$ iff $\forall x\ x \in A \wedge y < x \to y \in A$. A cut is proper if it is not equal to O.

Lemma 2. For all $f \in {}^{<\mathbb{N}}\{1,2\}$ is not order isomorphic to one of its proper cuts.

Proof. We prove this by induction on $\mathrm{dom}(f) \in \omega$.

If $\mathrm{dom}(f) = \emptyset$ there is nothing to prove.

Let $\mathrm{dom}(f) = n+1$. Let A be a proper cut of τ_f and φ be an order isomorphism from τ_f to A. By inductive hypothesis there is no order bijection from $\tau_{f\restriction n}$ to one of its proper cuts. First note that $\varphi[\tau_{f\restriction n}] = \tau_{f\restriction n}$. Indeed, since $\tau_{f\restriction n}$ is a cut of τ_f then $\varphi[\tau_{f\restriction n}]$ is also a cut of τ_f. Therefore, $\varphi[\tau_{f\restriction n}] \subseteq \tau_{f\restriction n}$ or $\tau_{f\restriction n} \subseteq \varphi[\tau_{f\restriction n}]$. If $\varphi[\tau_{f\restriction n}] \subset \tau_{f\restriction n}$ then $\varphi \restriction \tau_{f\restriction n}$ would be an order isomorphism from $\tau_{f\restriction n}$ to a proper cut of $\tau_{f\restriction n}$, which contradicts our inductive hypothesis. If on the other hand $\tau_{f\restriction n} \subset \varphi[\tau_{f\restriction n}]$ then $\varphi^{-1} \restriction \tau_{f\restriction n}$ would be again an order preserving isomorphism from $\tau_{f\restriction n}$ to a proper cut of $\tau_{f\restriction n}$, leading again to a contradiction. Therefore, $\varphi[\tau_{f\restriction n}] = \tau_{f\restriction n}$. Let us denote by ω^* the order type of the negative integers and by ζ the order type of the integers. Now we have four cases for the order type of $A \setminus \tau_{f\restriction n}$:

- $(A \setminus \tau_{f\restriction n}, <_{\tau_{f\restriction n}}) \cong \emptyset$: then trivially $(\tau_f \setminus A, <_{\tau_{f\restriction n}}) \cong f(n) + \zeta$. But this is a contradiction because $f(n) + \zeta$ is not order isomorphic to the empty order.

- $(A \setminus \tau_{f\upharpoonright n}, <_{\tau_{f\upharpoonright n}}) \cong f(n) + \omega^*$: note that then $(\tau_f \setminus A, <_{\tau_{f\upharpoonright n}}) \cong \omega$ and $\varphi[\tau_f \setminus A] = A \setminus \tau_{f\upharpoonright n}$. But it is easy to see that $f(n) + \omega^*$ is not order isomorphic to ω since $f(n) + \omega^*$ is not well ordered and ω is.

- $(A \setminus \tau_{f\upharpoonright n}, <_{\tau_{f\upharpoonright n}}) \cong 1$: then $(\tau_f \setminus A, <_{\tau_{f\upharpoonright n}}) \cong 1 + \zeta$ if $f(n) = 2$ or $(\tau_f \setminus A, <_{\tau_{f\upharpoonright n}}) \cong \zeta$ if $f(n) = 1$ but then trivially 1 is not order isomorphic to either $1 + \zeta$ or ζ. So we have a contradiction.

- $(A \setminus \tau_{f\upharpoonright n}, <_{\tau_{f\upharpoonright n}}) \cong 2$: as before $(\tau_f \setminus A, <_{\tau_{f\upharpoonright n}}) \cong \zeta$, so also in this case we get a contradiction.

Since in all the cases we reached a contradiction, then τ_f is not isomorphic to any of its proper cuts as desired. □

Now, let $f, g \in {}^{\mathbb{N}}\{1, 2\}$ be such that $f \neq g$. Let $n \in \mathbb{N}$ be the least such that $f(n) \neq g(n)$. Without loss of generality assume $f(n) = 2$ and $g(n) = 1$ Assume φ is an order isomorphism between τ_f and τ_g. By the previous lemma, $\varphi[\tau_{f\upharpoonright n}] = \tau_{f\upharpoonright n}$. Note that $\varphi(\langle 2n, 2\rangle) > \langle 2n, 1\rangle$. The set $\{z \in \tau_g \mid \langle 2n, 2\rangle < z < \varphi(\langle 2n, 2\rangle)\}$ is infinite. Indeed, $\varphi(\langle 2n, 2\rangle) = \langle m, m'\rangle$ for some $2n < m$ and $m' \in \mathbb{Z}$, therefore $\{\langle 2n, z\rangle \mid z < m', z \in \mathbb{Z}\}$ is a subset of $\{z \in \tau_g \mid \langle 2n, 2\rangle < z < \varphi(\langle 2n, 2\rangle)\}$. But this is a contradiction, because there is only one point in between $\tau_{f\upharpoonright n}$ and $\langle 2n, 2\rangle$, namely $\langle 2n, 1\rangle$. So φ cannot be an order isomorphism.

Finally, since we showed that the map sending every $f \in {}^{\mathbb{N}}\{1, 2\}$ to τ_f is an injection form ${}^{\mathbb{N}}\{1, 2\}$ to the set of order types of countable linear orders then we have that there are at least $\mathfrak{c}$ many such order types as desired. □

Exercise 3. Show that $\mathbb{Q}$ is not the intersection of countably many open sets.

Proof. Assume $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} O_n$ where for each $n \in \mathbb{N}$, we have that O_n is an open set. Note that, since for each $n \in \mathbb{N}$, we have $\mathbb{Q} \subset O_n$, then each O_n is open dense (in $\mathbb{R}$). Let $f : \mathbb{N} \to \mathbb{Q}$ be any enumeration of $\mathbb{Q}$. We have that for each $n \in \mathbb{N}$, the set $U_n := O_n \setminus \{f(n)\}$ is still open dense. Indeed, let $(a, b) \subset \mathbb{R}$, we have two cases: if $q \in (a, b)$, then (a, q) is an open set and $\emptyset \neq O_n \cap (a, q) = U_n \cap (a, q)$; if $q \notin (a, b)$ then $\emptyset \neq O_n \cap (a, b) = U_n \cap (a, b)$. So for each $n \in \mathbb{N}$ the set U_n is dense. To show that it is also open it is enough to see that if $x \in U_n$, then there is $(a, b) \subseteq O_n$ such that $x \in (a, b)$, if $q \notin (a, b)$ then we have done since $(a, b) \subset U_n$, otherwise either $x \in (a, q) \subseteq U_n$ or $x \in (q, b) \subseteq U_n$, in both cases there is an open interval containing x completely contained in U_n.

Now note that $\bigcap_{n\in\mathbb{N}} U_n$ is a countable intersection of dense open sets so by the Baire category theorem it is not empty. But this is a contradiction since $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} O_n \setminus \{f(n)\} = \bigcap_{n\in\mathbb{N}} O_n \setminus \bigcup_{n\in\mathbb{N}}\{f(n)\} = \mathbb{Q} \setminus \mathbb{Q} = \emptyset$. □

Exercise 4. Given a set X of real numbers we define by transfinite recursion the following sequence:

$$\begin{aligned} X_0 &= X, \\ X_{\alpha+1} &= X'_\alpha, \\ X_\lambda &= \bigcap_{\alpha<\lambda} X_\alpha \text{ for } \lambda \text{ limit.} \end{aligned}$$

We call *Cantor-Bendixson rank* of X the smallest ordinal α such that $X_\alpha = X_{\alpha+1}$. Give examples of sets X such that:

(a) X has Cantor-Bendixson rank 2;

(b) X has Cantor-Bendixson rank 3.

Proof. Consider the set $X = \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$. Note that since $\lim_{n\to\infty} \frac{1}{n} = 0$, then 0 is a limit point of X. Moreover every point <0 is not a limit point of X since every point of the sequence is strictly bigger than 0. If $1 \geq r > 0$ then let $n \in \mathbb{N}$ such that $n \leq \frac{1}{r} < n + 1$ then $\frac{1}{n+1} < r \leq \frac{1}{n}$ let $m = \min\{\frac{1}{n} - r, r - \frac{1}{n+1}\}$ then the interval $(r - m, r + m)$ will contain at most one point of X namely r. If $r > 1$ the it is trivially isolated since the interval $(r - \frac{r-1}{2}, r + \frac{r-1}{2})$ does not contain any point of X.

Therefore we have $X_0 = X$, $X_1 = \{0\}$. Finally since 0 is trivially isolated in X_1 we get $X_2 = \emptyset = X_3$.

We want to define a set X of rank 3. Note that for every interval $(a, b) \subset \mathbb{R}$, the map $f_{(a,b)}(x) := \frac{x-a}{b-a}$ is an order isomorphism from $[a, b]$ to $[0, 1]$. Trivially this map preserves convergency, i.e., if x is a limit point for $Y \subset [a, b]$ iff $f(x)$ is a limit point of $f[Y]$. For each $n \in \mathbb{N}^+$, let $f_n := f_{(\frac{1}{n+1}, \frac{1}{n})}$ and $f_0 := id$.

For each $n \in \mathbb{N}$, define $S_n := \{f_n^{-1}(\frac{1}{m+1}) \mid m \in \mathbb{N}\}$. Take $X_0 := \bigcup_{n\in\mathbb{N}} S_n$. Note that, since for each $n, m \in \mathbb{N}$ such that $m \neq n$ we have that $S_n \subset (\frac{1}{n+1}, \frac{1}{n})$, $S_m \subset (\frac{1}{m+1}, \frac{1}{m})$ and $(\frac{1}{n+1}, \frac{1}{n}) \cap (\frac{1}{m+1}, \frac{1}{m}) = \emptyset$, the set of limit points of X_0 is $\bigcup_{n\in\mathbb{N}} S_n'$. Since each S_n is an isomorphic copy of X then the previous proof shows that each S_n has exactly one isolated point namely $f_n^{-1}(0)$. Then we have that $X_1 = \{f_n^{-1}(0) \mid n \in \mathbb{N}\} = \{0\} \cup \{\frac{1}{n+1} \mid n \in \mathbb{N}\}$ and we already showed in (a) that $X_2 = 0$ and $X_3 = \emptyset = X_4$ as desired.

□