HOMEWORK 6

SET THEORY

▶ **1**. Let κ, λ, and μ be cardinal numbers. Show that

(a) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$;

(b) $\kappa^{(\lambda+\mu)} = \kappa^\lambda \cdot \kappa^\mu$;

(c) $(\kappa^\lambda)^\mu = \kappa^{(\lambda\cdot\mu)}$;

(d) if $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$; and

(e) if $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.

▶ **2**. For a set X, we denote by $X!$ the set of all bijections from X to X.

(a) Show that for any sets X and Y, if $|X| = |Y|$, then $|X!| = |Y!|$.

(b) Show that if α is an infinite ordinal, then $|\alpha!| = |\wp(\alpha)|$.

▶ **3**. Let α be a limit ordinal. A subset $X \subseteq \alpha$ is called *cofinal* if for every $\beta \in \alpha$ there is a $\gamma > \beta$ such that $\gamma \in X$. If δ is any ordinal and $f : \delta \to \alpha$ is a function, we say that f is *cofinal* if $\mathrm{ran}(f)$ is cofinal. In class, we defined $\mathrm{cf}(\alpha) := \min\{\kappa\,;$ there is a cofinal set $X \subseteq \alpha$ such that $\kappa = |X|\}$. Show that the following three cardinal numbers are the same:

(i) $\mathrm{cf}(\alpha)$;

(ii) $\min\{\beta\,;$ there is a cofinal function $f : \beta \to \alpha\}$;

(iii) $\min\{\beta\,;$ there is an order-preserving cofinal function $f : \beta \to \alpha\}$.

▶ **4**. Show that the cardinality of $\wp(\mathbb{N})$ cannot be $\aleph_\omega$. [You are not allowed to use Jech's Theorem 3.11 since we didn't prove it in class, but looking at the proof of the theorem in the book may give you an idea how to approach this.]

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