HOMEWORK 4

SET THEORY

Since ordinals are linear orders with respect to $\in$, we often write $<$ for the element relation, i.e., "$\alpha < \beta$" stands for $\alpha \in \beta$ if α and β are ordinals.

► **1**. A function f with $\mathrm{dom}(f) = \mathbb{N}$ is called an *infinite descending sequence* if for all $n \in \mathbb{N}$ we have $f(n+1) \in f(n)$. Show that there is no infinite descending sequence such that all elements of the sequence are ordinals.

► **2** (Jech: 2.6). Show that for every ordinal α there is a $\beta > \alpha$ such that β is a limit ordinal.

► **3** (Jech: 2.7). We call a class F a *sequence of ordinals* if

(1) all elements of F are pairs (α, β) of ordinals,
(2) F is *functional* in the sense that if $(\alpha, \beta) \in F$ and $(\alpha, \beta') \in F$, then $\beta = \beta'$, and
(3) F is *total* in the sense that for each ordinal α there is a β such that $(\alpha, \beta) \in F$.

As usual, we write $F(\alpha)$ for the unique β such that $(\alpha, \beta) \in F$. An ordinal ξ is called a *fixed point of* F if $F(\xi) = \xi$. A sequence of ordinals is called *normal* if

(1) for $\alpha < \beta$, $F(\alpha) < F(\beta)$ and
(2) for limit ordinals λ, $F(\lambda) = \bigcup\{F(\alpha) : \alpha < \lambda\}$.

Show that every normal sequence of ordinals has arbitrarily large fixed points, i.e., for every α there is $\xi > \alpha$ such that ξ is a fixed point of F.

► **4** (Jech: parts of Lemmas 2.21 & 2.25). Prove the following statements for ordinals α, β, and γ:

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$;
(b) if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$;
(c) if $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$;
(d) if $\alpha < \beta$, then there is a unique δ such that $\alpha + \delta = \beta$; and
(e) if $\alpha > 0$ and γ is arbitrary, then there is a unique pair (β, ϱ) with the properties $\varrho < \alpha$ and $\gamma = \alpha \cdot \beta + \varrho$.

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