Hessenberg's Theorem (ZFC). If X is infinite, then $X \times X \sim X$.

Remark 1. For ordinals $\alpha \geq \omega$, one can show $\alpha \times \alpha \sim \alpha$ in ZF, i.e., without using the Axiom of Choice, as was done in Question 3 of Homework 5. This result immediately implies the above theorem (since AC implies that every set is in bijection with some ordinal). The given proof here is different from the constructive proof from Homework 5.

Proof. In ZFC, every infinite set is in bijection with some aleph $\aleph_\alpha$, so it is enough to show that $\aleph_\alpha \times \aleph_\alpha \sim \aleph_\alpha$ for all α. We shall show that by induction on α.

Fix α, a set X with $|X| = \aleph_\alpha$, and suppose that for all $\gamma < \alpha$, we have that $\aleph_\gamma \times \aleph_\gamma \sim \aleph_\gamma$.

Consider the set $H := \{f\,;\, f \text{ is a bijection between } Z \times Z \text{ and } Z \text{ for some infinite subset } Z \subseteq X\}$, ordered by inclusion. If $C \subseteq H$ is a chain, then consider $f_C := \bigcup\{f\,;\, f \in C\}$ and let $Z_C := \bigcup\{\mathrm{ran}(f)\,;\, f \in C\}$. It is easy to see that f_C is a bijection between $Z_C \times Z_C$ and Z_C, so $f_C \in H$ and f_C is an upper bound of C. Thus, $(H, \subseteq)$ is a chain complete partial order. Consequently, by Zorn's Lemma, $(H, \subseteq)$ has a maximal element h with $Z_h := \mathrm{ran}(h)$.

If $|Z_h| = \aleph_\alpha$, then h witnesses that $\aleph_\alpha \times \aleph_\alpha \sim \aleph_\alpha$ and we are done. Let us therefore assume that $|Z_h| = \aleph_\gamma < \aleph_\alpha$ and derive a contradiction.

Claim. $|X \backslash Z_h| < \aleph_\gamma$.

[Otherwise, we find $D \subseteq X \backslash Z_h$ of cardinality $\aleph_\gamma$. Then $Y := D \cup Z_h$ has cardinality $\aleph_\gamma$ and

$$Y \times Y = (Z_h \times Z_h) \cup (D \times Z_h) \cup (Z_h \times D) \cup (D \times D).$$

We observe

$$\begin{aligned} \aleph_\gamma &\leq |(D \times Z_h) \cup (Z_h \times D) \cup (D \times D)| \\ &\leq |\aleph_\gamma \times \aleph_\gamma| + |\aleph_\gamma \times \aleph_\gamma| + |\aleph_\gamma \times \aleph_\gamma| \\ &= \aleph_\gamma + \aleph_\gamma + \aleph_\gamma \text{ (induction hypothesis)} \\ &= \aleph_\gamma, \end{aligned}$$

so there is a bijection $g : (D \times Z_h) \cup (Z_h \times D) \cup (D \times D) \to D$. The functions g and h have disjoint domains and ranges, so combine g and h to a bijection between $Y \times Y$ and Y which extends h in contradiction to the maximality of h.]

The claim now implies that $|X| = |(X \backslash Z_h) \cup Z_h| \leq |X \backslash Z_h| + |Z_h| \leq \aleph_\gamma$ in contradiction to the assumption that $|X| = \aleph_\alpha > \aleph_\gamma$. q.e.d.

Remark 2. Note that the proof does not show that the maximal element h of H provided by Zorn's Lemma is a bijection between $X \times X$ and X, only that the size of the range of h is the same as the size of X. Suppose $|X| = \kappa \geq \omega$, $x \in X$, and suppose we know that $X \times X \sim X$. Then $Y := X \backslash \{x\}$ still has cardinality κ, so $Y \times Y \sim Y$. Let b be any bijection witnessing that. Then b cannot be extended: the only way to extend it would be to $X \times X$, but this means that we need to add κ many elements to the domain and exactly one element to the range which cannot be done while preserving injectivity. Thus, b is a maximal element in H, but $\mathrm{ran}(b) \neq X$.