Solutions Set Theory HW1

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In this text you will find solutions to the exercises of the first homework of Set Theory.

Exercise 1. $(\forall a)(\forall b)(\forall c)(\forall d)\big((a,b)=(c,d) \Leftrightarrow a=c \wedge b=d\big)$.

Proof. From right to left. Let us assume $a=c$ and $b=d$, we need to show $(a,b)=(c,d)$. Since $a=c$ by *extensionality* we have that

$$\{a\}=\{c\}. \tag{1}$$

Similarly, since $a=c$ and $b=d$, again by *extensionality* we have

$$\{a,b\}=\{c,d\}. \tag{2}$$

By definition $(a,b)=\big\{\{a\},\{a,b\}\big\}$ and $(c,d)=\big\{\{c\},\{c,d\}\big\}$. Therefore, by (1), (2) and *extensionality* we get $(a,b)=(c,d)$ as desired.

For the other direction assume $(a,b)=(c,d)$. We have two cases:

$a=b$**:** by definition $(a,b)=\big\{\{a\},\{a,b\}\big\}$. Since $a=b$ by *extensionality* $\{a\}=\{a,b\}$. By applying *extensionality* again we have $\big\{\{a\},\{a,b\}\big\}=\big\{\{a\}\big\}$. Now, by definition $(c,d)=\big\{\{c\},\{c,d\}\big\}$ and since we assumed $(a,b)=(c,d)$, we have $\big\{\{c\},\{c,d\}\big\}=\big\{\{a\}\big\}$. But then $\{c\}=\{a\}$ and $\{c,d\}=\{a\}$ which imply $d=a=b$ and therefore $d=b$.

$a\neq b$**:** by the definition of ordered pair and our assumption we have $\big\{\{a\},\{a,b\}\big\}=\big\{\{c\},\{c,d\}\big\}$. By reverse extensionality we have:

$$(\forall x)\big(x\in(a,b)\Leftrightarrow x\in(c,d)\big)$$

Note that since $a\neq b$ we have $\{a\}\neq\{a,b\}$. Therefore we have the following cases:

- If $\{a\}=\{c,d\}$ and $\{a,b\}=\{c\}$: then $c=a=d$ and $b=c=a$ which is a contradiction. Therefore this case is impossible.
- If $\{a\}=\{c\}$ and $\{a,b\}=\{c,d\}$: then $c=a$. Since $a\neq b$ we also have $b=d$ as desired.

Alternatively one starts from $(a,b)=(c,d)$ and concludes that both $\{a\}$ and $\{a,b\}$ belong to $\big\{\{c\},\{c,d\}\big\}$. Hence $\{a\}=\{c\}$ or $\{a\}=\{c,d\}$.

It the first case one has $a=c$ and one then considers whether $\{a,b\}=\{c\}$ or $\{a,b\}\neq\{c\}$. In the former case one has $b=c$ as well and hence $b=a$ so that $(a,b)=\big\{\{a\}\big\}$ but then $\{c,d\}\in\big\{\{a\}\big\}$, hence $\{c,d\}=\{a\}$ and so $d=a$ as well. In the latter case $\{a,b\}\neq\{a\}$ as well, so $b\neq a$; also we must have $\{a,b\}=\{c,d\}=\{a,d\}$, so $b\in\{a,d\}$ and therefore $b=d$.

The second case, $\{a\}=\{c,d\}$ leads as above to $c=d=a$ and then to $b=a$ as well. □

Exercise 2. Prove that there is no set X such that $\mathcal{P}(X)\subseteq X$.

Proof. Assume that X is such that $P(X)\subseteq X$. By *separation* define the following set:

$$S=\{x\in X : x\notin x\}$$

Since for all $x\in S$ we have $x\in X$, the set S is a subset of X. Therefore, by definition $S\in\mathcal{P}(X)$ and by our assumption $S\in X$. But this is a contradiction since by definition $S\in S\Leftrightarrow S\notin S$. Therefore there is no X such that $\mathcal{P}(X)\subseteq X$.

Alternatively one can take X arbitrary and define S as above and then prove, possibly by contradiction, that $S\in\mathcal{P}(X)\setminus X$. □

Exercise 3. Verify: if $a \in A$ and $b \in B$ then $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$ and $a, b \in \bigcup(a, b)$.

Proof. Assume $a \in A$ and $b \in B$. By the definition $A \cup B = \bigcup\{A, B\}$. By the *Axiom of union* $a \in \bigcup\{A, B\}$ and $b \in \bigcup\{A, B\}$. Therefore, by the *definition* of subset we have $\{a\} \subset A \cup B$ and $\{a, b\} \subset A \cup B$. Now, by the *Power set axiom* we get $\{a, b\} \in \mathcal{P}(A \cup B)$ and $\{a\} \in \mathcal{P}(A \cup B)$. Again by the *definition* of subset, we have $\big\{\{a\}, \{a, b\}\big\} \subset \mathcal{P}(A \cup B)$ which by the *Power set axiom* implies $\big\{\{a\}, \{a, b\}\big\} \in \mathcal{P}(\mathcal{P}(A \cup B))$. Finally by definition $\big\{\{a\}, \{a, b\}\big\} = (a, b)$, and therefore $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$.

Since, by definition, $a \in \{a\}$, $b \in \{a, b\}$ and both $\{a\}$ and $\{a, b\}$ are in (a, b), by the *Axiom of union* we have $a, b \in \bigcup(a, b)$ as desired. ☐

Exercise 4. Write out in full (no abbreviations) a formula that expresses

1. "z is an ordered pair".
2. "z is an ordered pair and x is its first coordinate".

Proof. First let us define the following two auxiliary formulas:

$$\varphi(x, a) := (\forall y)(y \in x \Leftrightarrow y = a), \text{ i.e., } x \text{ is the singleton of } a,$$

and

$$\varphi'(x, a, b) := (\forall y)(y \in x \Leftrightarrow y = a \vee y = b), \text{ i.e., } x \text{ is the pair } \{a, b\}$$

Now, for 1 we define:

$$\xi(z) := (\exists a)(\exists b)(\forall x)(x \in z \Leftrightarrow \varphi(x, a) \vee \varphi'(x, a, b))$$

For 2 we define:

$$\xi'(z, x) := (\exists b)(\forall y)(y \in z \Leftrightarrow \varphi(y, x) \vee \varphi'(y, x, b)) \quad \square$$