HOMEWORK 8

SET THEORY

► **1** (Jech: 3.14, 5.6 & 5.7). Remember that a set D was called *Dedekind finite* if it has no proper subset that is equinumerous to it. Show (in ZF) that the following is a chain of implications, i.e., each of the statements in the list implies those that are further down in the list:

(i) AC;
(ii) DC;
(iii) AC_ω;
(iv) every infinite set has a countable subset;
(v) every Dedekind finite set is finite.

► **2**. A topological space (X, τ) is called *compact* if every open covering of X (i.e., any $C \subseteq \tau$ such that $\bigcup C = X$) has a finite subset C_0 such that $\bigcup C_0 = X$. Tychonoff's theorem, denoted by TT, says: "If I is any non-empty set and for each $i \in I$, (X_i, τ_i) is a compact topological space, then the product space $\prod_{i \in I}(X_i, \tau_i)$ is compact". Show that in ZF, TT implies AC.

[*Hint.* If S is any family of non-empty sets with index set I, let $\infty \notin \bigcup_{i \in I} S(i)$, and consider $T(i) := S(i) \cup \{\infty\}$ with the topology $\tau_i := \{\varnothing, \{\infty\}, S(i), T(i)\}$. Apply TT to the family $(T(i), \tau_i)$ to show that the family consisting of $X_i := \{f\,;\, f(i) = \infty\}$ is not an open covering of the product.]

► **3**. Show in ZFC that the following are equivalent:

(i) $2^{\aleph_0} = \aleph_1$ and
(ii) every set of real numbers is either countable or equinumerous to the set of all real numbers.

(Which of the two directions does not need a use of the Axiom of Choice?)

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