HOMEWORK 3

► **1**. Suppose X is a set, $R \subseteq X \times X$ a binary relation on X and $\sim$ is an equivalence relation on X. As usual, we write $X/{\sim}$ for the set of $\sim$-equivalence classes and we write $[x]_\sim := \{y \in X : y \sim x\}$. We say that $\sim$ is a *congruence relation with respect to* R if the following holds: if $x\ R\ y$, and if $x \sim x'$ and $y \sim y'$, then $x'\ R\ y'$. If $\sim$ is a congruence with respect to R, then we can define R on $X/{\sim}$ on the basis of representatives, i.e., $[x]_\sim\ R\ [y]_\sim$ if and only if $x\ R\ y$.

Let (X, R) be a preorder and define a relation $\equiv$ on X by $x \equiv y$ if and only if $x\ R\ y$ and $y\ R\ x$.

(a) Show that $\equiv$ is an equivalence relation on X which is a congruence relation with respect to R.
(b) Show that $(X/{\equiv}, R)$ is a partial order.
(c) Show that if (X, R) is a linear preorder, then $(X/{\equiv}, R)$ is a linear order.
(d) Show that if (X, R) is well-founded, then so is $(X/{\equiv}, R)$.

► **2**. Let X be a set and consider the relation R defined on X by $x\ R\ y$ if and only if $x \subsetneq y$. Give examples of sets X with the following properties:

(a) X is infinite and transitive, and R is a linear order,
(b) X is infinite and not transitive, and R is a linear order,
(c) X is inductive and R is not a linear order.

► **3**. We say that a structure $(X, <, s, z)$ is a *Peano structure* if $<$ is a binary relation on X, s is a function from X to X and z is an element of X. A set $A \subseteq X$ is *closed under* s if for all x, if $x \in A$, then $s(x) \in A$.

We say that a Peano structure $(X, <, s, z)$ satisfies the *principle of complete induction* if the following holds: if $A \subseteq X$ satisfies

- $z \in A$ and
- A is closed under s

then $A = X$.

We say that a Peano structure $(X, <, s, z)$ satisfies the *principle of order induction* if the following holds: if $A \subseteq X$ satisfies

- $z \in A$ and
- if $\{y \in X : y < x\} \subseteq A$ then $x \in A$

then $A = X$.

Let α be an ordinal, then we can define a *successor* function s by $s(\gamma) := \gamma \cup \{\gamma\}$. If α is an ordinal closed under s and $0 \in \alpha$, then $(\alpha, \in, s, 0)$ is a Peano structure.

(a) Show that for every ordinal α which is closed under s and such that $0 \in \alpha$, the Peano structure $(\alpha, \in, s, 0)$ satisfies the principle of order induction.
(b) Show that for an ordinal α which is closed under s and such that $0 \in \alpha$, the Peano structure $(\alpha, \in, s, 0)$ satisfies the principle of complete induction if and only if $\alpha = \omega$.

► **4**. If (X, R) and (Y, S) are strict linear orders, we define the *order sum* $(X, R) \oplus (Y, S)$ as follows: we let $Z := (\{0\} \times X) \cup (\{1\} \times Y)$ and define a relation T on Z by $(b, z)\ T\ (c, z')$ if and only if either $b < c$ or ($b = c = 0$ and $z\ R\ z'$) or ($b = c = 1$ and $z\ S\ z'$). The order sum is then (Z, T).

Show that

(a) $(X, R) \oplus (Y, S)$ is a strict linear order and
(b) if both (X, R) and (Y, S) are well-founded, then so is $(X, R) \oplus (Y, S)$.

► **5**. Consider $(Z, T) := (\mathbb{N}, <) \oplus (\mathbb{N}, <)$, the order sum of two copies of the natural numbers (as in Problem 4). By Problem 4 (b), this is a wellorder.

(a) Does $(\mathbb{N}, <) \oplus (\mathbb{N}, <)$ have a maximal element?
(b) Define an operation $s : Z \to Z$ and some z in Z such that (Z, T, s, z) becomes a Peano structure and $s(x)$ is the T-least element bigger than x. Does this structure satisfy the principle of complete induction?

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