Homework 2: some solutions from your fellow students.

We should like to thank Adrien Champougny, Raja Damanik and Romain Grausi for sending us the LaTeX code of their solutions for homework 2. In this file, you can find slightly modified versions of their solutions which received very high marks and therefore can be considered as models for what solutions to the questions should look like. Of course, there is not a unique solution to each question.

1. Prove: The Separation Axiom follows from The Replacement Axiom.

   Solution.

   Let $\varphi(u,p)$ be a formula, X and P. We want to show that $Y = \{x : x \in X \wedge \varphi(x,P)\}$ is a set (using replacement axiom).

   Let $\psi(x,y,p) := \varphi(x,p) \wedge (y = x)$. Indeed, if $\psi(x,y,p)$ and $\psi(x,z,p)$ hold, then $y = x$ and $z = x$, and hence $y = z$. Hence $F = \{(x,y) : \psi(x,y,P)\}$ is a class function.

   By replacement axiom, $F[X]$ is a set. Note that

$$\begin{aligned} t \in F[X] &\Leftrightarrow (\exists x \in X)(F(x) = t) \\ &\Leftrightarrow (\exists x \in X)(\varphi(x,P) \wedge (x = t)) \\ &\Leftrightarrow t \in X \wedge \varphi(t,P) \\ &\Leftrightarrow t \in Y, \end{aligned}$$

   and hence by extensionality, $Y = F[X]$. Hence Y is a set.

2. Use the separation schema to prove the union axiom, the power set axiom, and the replacement axioms from their weaker version:

   (1) $(\forall X)(\exists Y)(\bigcup X \subseteq Y)$ or rather $(\forall X)(\exists Y)(\forall x \in X)(\forall u)(u \in x \to u \in Y)$.

   (2) $(\forall X)(\exists Y)(\mathcal{P}(X) \subseteq Y)$ or rather $(\forall X)(\exists Y)(\forall u)(u \subseteq X \to u \in Y)$.

   (3) if F is a class function then $(\forall X)(\exists Y)(F(X) \subseteq Y)$.

   Solution.

   (1) Let X be a set. We want to show $\bigcup X = \{u : (\exists x)(x \in X \wedge u \in x)\}$. By weaker version of union axiom, there is a set Y such that $\bigcup X \subseteq Y$. By separation axiom, then we have a set $U = \{u \in Y : (\exists x)(x \in X \wedge u \in x)\}$. We show that $U = \bigcup X$.
   Let $t \in U$. By definition of U, $(\exists x)(x \in X \wedge t \in x)$. Hence, $t \in \bigcup X$.
   Let $t \in \bigcup X$. Since $\bigcup X \subseteq Y$, then $t \in Y$. Moreover by definition of U, $(\exists x)(x \in X \wedge t \in x)$. Since $t \in Y$ and $(\exists x)(x \in X \wedge t \in x)$, then $t \in U$.
   Since $(\forall t)(t \in U \leftrightarrow t \in \bigcup X)$, by extensionality axiom, $U = \bigcup X$.

(2) Let X be a set. We want to show $\mathcal{P}(X) = \{x : x \subseteq X\}$ is a set. By weaker version of power set axiom, there is a set Y such that $\mathcal{P}(X) \subseteq Y$. By separation axiom, then we have a set $P = \{p \in Y : p \subseteq X\}$. We show that $P = \mathcal{P}(X)$.

Let $t \in P$. By definition of P, $t \subseteq X$, and hence $t \in \mathcal{P}(X)$.

Let $t \in \mathcal{P}(X)$. Since $\mathcal{P}(X) \subseteq Y$, $t \in Y$. Moreover by definition of $\mathcal{P}(X)$, $t \subseteq X$. Since $t \in Y$ and $t \subseteq X$, then $t \in P$.

Since $(\forall t)(t \in P \leftrightarrow t \in \mathcal{P}(x))$, by extensionality axiom, $P = \mathcal{P}(X)$.

(3) Let F be a class function and X be a set. We want to show that $F(X) = \{y : (\exists x \in X)(F(x) = y)\}$ is a set. By weaker version of replacement axiom, there is a set Y such that $F(X) \subseteq Y$. By separation axiom, then we have a set $Z = \{y \in Y : (\exists x \in X)(F(x) = y)\}$. We show that $Z = F(X)$.

Let $t \in Z$. By definition of Z, $(\exists x \in X)(F(x) = t)$, and hence $t \in F(X)$.

Let $t \in F(X)$. Since $F(X) \subseteq Y$, then $t \in Y$. Moreover, by definition of $F(X)$, $(\exists x \in X)(F(x) = t)$. Since $t \in Y$ and $(\exists x \in X)(F(x) = t)$, it follows that $t \in Z$.

Since $(\forall t)(t \in Z \leftrightarrow t \in F(X))$, by extensionality axiom, $Z = F(X)$.

3. Prove the following about inductive sets: if X is an inductive set, then so are:

(1) $\{x \in X : x \subseteq X\}$.

(2) $\{x \in X : x \,\text{is transitive}\}$.

(3) $\{x \in X : x \,\text{is transitive and}\, x \notin x\}$.

(4) $\{x \in X : x \,\text{is transitive and every non-empty set of}\, x \,\text{has}\, \in -\text{minimal element}\}$.

(5) $\{x \in X : x = \emptyset \,\text{or}\, x = y \cup \{y\} \,\text{for some}\, y\}$.

Solution.

(1) Let Y be such set. First, we show $\emptyset \in Y$. Since X is inductive, $\emptyset \in X$. Moreover, since $\emptyset \subseteq X$, we then have $\emptyset \in Y$.

Suppose $y \in Y$. Then

- $y \cup \{y\} \in X$ because $y \in X$ and X inductive.
- $y \cup \{y\} \subseteq X$.
  Let $t \in y \cup \{y\}$.
  Suppose $t \in y$. Since $y \in Y$, then by definition of Y, $y \subseteq X$. Hence $t \in X$.
  Suppose $t \in \{y\}$. Hence, $t = y$. By definition of Y, $y \in X$, and hence $t \in X$.

and hence $y \cup \{y\} \in Y$.

Hence, Y is an inductive set.

(2) Let Y be such set. First we show $\emptyset \in Y$. Since X is inductive $\emptyset \in X$. Moroeever, $\emptyset$ is transitive (there is no element of $\emptyset$ so there is nothing to satisfy). Hence $\emptyset \in Y$.

Suppose $y \in Y$. We show that $y \cup \{y\} \in Y$. This is true because:

- $y \cup \{y\} \in X$, same as in (1).
- $y \cup \{y\}$ is transitive.
  Let $t \in y \cup \{y\}$. We show that $t \subseteq y \cup \{y\}$.
  Suppose $t \in y$. Since y is transitive, $t \subseteq y$ and hence $t \subseteq y \cup \{y\}$.
  Suppose $t \in \{y\}$. Hence, $t = y$, but then $t \subseteq y$ and consequently $t \subseteq y \cup \{y\}$.

Hence, Y is also inductive.

(3) Let Y be such set.

First we show that $\emptyset \in Y$. Since X is inductive, $\emptyset \in X$. Moreover, $\emptyset$ is transitive and since it has no element. Finally, $\emptyset \notin \emptyset$ also since $\emptyset$ does not have any element. Thus, $\emptyset \in Y$.

Suppose $y \in Y$. We show $y \cup \{y\} \in Y$. This is true because:

- $y \cup \{y\} \in X$ and $y \cup \{y\}$ is transitive, same as in (2).
- $y \cup \{y\} \notin y \cup \{y\}$.
  Since $y \in Y$, then y is transitive and $y \notin y$. $(*)$
  Suppose for the sake of contradiction $y \cup \{y\} \in y$. Then, by transitivity of y, $y \cup \{y\} \subseteq y$. Hence for all t, if $t \in y \cup \{y\}$, then $t \in y$. In particular, since $y \in y \cup \{y\}$, then $y \in y$, contradicting $(*)$. Hence, $y \cup \{y\} \notin y$.
  Suppose for the sake of contradiction again, $y \cup \{y\} \in \{y\}$. Hence, $y \cup \{y\} = y$. By extensionality axiom, for all t, we have $t \in y \vee t \in \{y\} \leftrightarrow t \in y$. In particular, for $t = y$, since $y \in \{y\}$, we have $y \in y$, again contradicting $(*)$. Hence, $y \cup \{y\} \notin \{y\}$.
  Since $y \cup \{y\} \notin y$ and $y \cup \{y\} \notin \{y\}$, then $y \cup \{y\} \notin y \cup \{y\}$.

(4) Let Y be such set.

First we show that $\emptyset \in Y$. Since X is inductive, $\emptyset \in X$. Moreover, $\emptyset$ is transitive and since it has no element. Hence, $\emptyset \in X$.

Suppose $y \in Y$. We show $y \cup \{y\} \in Y$. First note that $y \notin y$. Indeed, assume $y \in y$, since $\{y\} \subseteq y$ and $y \in Y$, we have that $y \notin y$ which contradicts our hypothesis.

Now, let $z \subseteq y \cup \{y\}$ be such that $z \neq \emptyset$. We have two cases:

- if $y \notin z$ then $z \subseteq y$ and z has an $\in$-minimal element since $y \in Y$.
- if $y \in z$ then consider the set $z' := z \cap y$. If $z' = \emptyset$ then $z = \{y\}$. Since, as we proved $y \notin y$, then y is $\in$-minimal in $\{y\}$. Finally, if $z' \neq \emptyset$, since $z' \subseteq y$ and $y \in Y$, there is $m \in z'$ such that m is $\in$-minimal in z'. We claim that m is $\in$-minimal in z. To prove this, it is enough to show that $y \notin m$ because $z' = z \cap y$ and $z' \subseteq y \cup \{y\}$. Assume by contradiction that $y \in m$. Since $m \in z' \subset y$ then $m \in y$, therefore by transistivity of y we have $y \in y$ but this is a contradiction since as we proved $y \notin y$.

(5) Let Y be such set.

Since $\emptyset \in X$ and by definition of Y itself, $\emptyset \in Y$.

Suppose $t \in Y$. Then $t \in X$ and since X is inductive, $t \cup \{t\} \in X$. Moreover, since $t \cup \{t\}$ is of the form $y \cup \{y\}$ by taking y as t, hence $t \cup \{t\} \in Y$ also.

Hence, Y is an inductive set.

4. We use infinity axiom to show that $\mathbb{N}$ is a set, the smallest inductive set.

(1) Here, we define $P(x)$ be the property that says "A^x is a set" (A^x is class of all functions from x to A). There is first order formula that corresponds to $P(x)$. By separation axiom, $S = \{n \in \mathbb{N} : P(n)\}$ is a set. We prove that S is inductive.

Note that $\emptyset \in S$ since $A^\emptyset = \{\emptyset\}$ is a set by pairing axiom.

Now suppose $n \in S$, hence there is a set A^n. We show that A^{n+1} is a set.

Let $\varphi(x, y)$ be a formula defined as $(\exists a)(\exists b)(x = (a, b) \wedge y = a \cup \{b\})$. If $\varphi(x, y)$ and $\varphi(x, z)$ hold, then $x = (a, b)$ for some a, b and $y = a \cup \{b\}$ and $z = a \cup \{b\}$. This implies $y = z$. Let $F = \{(x, y) : \varphi(x, y)\}$ be a class function.

Note that by pairing axiom, $\{n\}$ is a set and we have shown in class that product of two sets is indeed a set. Thus, $\{n\} \times A = \{(n, a) : a \in A\}$ is a set. Let X_n denotes $\{n\} \times A$. Moreover, $A^n \times X_n$ is a set.

Finally, by replacement axiom, $F[A^n \times X_n]$ is a set. Note that $f \in F[A^n \times X_n]$ iff there is a function $g \in A^n$ and a pair $(n, a) \in X_n$ such that $f = g \cup \{(n, a)\}$ iff $f \in A^{n+1}$. Hence, $A^{n+1} = F[A^n \times X_n]$ is indeed a set.

Now, since S is inductive, then $\mathbb{N} \subseteq S$ since $\mathbb{N}$ is the smallest inductive set, i.e. every $n \in \mathbb{N}$ is in S, and thus every n has property P: there exists a set A^n.

(2) Let $\psi(x, y, p)$ be a formula that says "there is u such that $y = (x, u)$ where for all s, we have $s \in u$ if and only if s is a function from x to p". By saying a function from x to p we mean s is a function, $\mathrm{dom}(s) = x$, and $\mathrm{ran}(s) \subseteq p$.

Suppose $\psi(x, y, p)$ and $\psi(x, z, p)$. Then there are u and v such that $y = (x, u)$ and $z = (x, v)$ such that for all s, $s \in u$ iff s is a function from x to p iff $s \in v$; hence $u = v$ and consequently $y = z$. Now, we can define a class function $F = \{(x, y) : \varphi(x, y, A)\}$. Note that, $F(x) = (x, A^x)$.

Since by (1) A^n is a set, (n, A^n) is also a set by the pairing axiom. Hence, $F(n) = (n, A^n)$ for each $n \in \mathbb{N}$.

By infinity axiom, there is an inductive set and we can take $\mathbb{N}$ as the smallest inductive set. Now, by replacement axiom, $F[\mathbb{N}] = \{(n, A^n) : n \in \mathbb{N}\}$ is a set.

(3) Let $H = F[\mathbb{N}]$ that we have constructed in (2). It is easy to see that H is a function; in particular for $n \in \mathbb{N}$, $H(n) = A^n$. By replacement schema, since $\mathbb{N}$ is a set, then $H[\mathbb{N}]$ is a set, i.e. $\{A^n : n \in \mathbb{N}\}$ is a set. Now, by union axiom, we have that $\bigcup\{A^n : n \in \mathbb{N}\}$ is a set.

(4) Let $F = \{(x, y) : \phi(x, y)\}$ be a class function defined by the formula ϕ such that $\phi(x, y) : (\forall c)(c \in y \leftrightarrow (\exists b)(b, c) \in x)$. This is a class function because if $\phi(x, y)$ and $\phi(x, z)$ hold then for all c, we have $c \in y \leftrightarrow (\exists b)(b, c) \in x \leftrightarrow c \in z$ and hence $y = z$ by extensionality axiom.

Since A^n is a set by (1) and F is a class function, we have $F[A^n]$ is a set. Note that if $y \in F[A^n]$ then y is the range of x for some $x \in A^n$. In particular, $y \subseteq A$ that and there is a function from n to y.

So, let $\psi(x, n)$ be the formula that says "there is a bijection with from x to n"; there is a first order formula that corresponds to this because we have formulas for the property one-to-one and onto of a function from x to n. Then by separation axiom, $\{x \in F[A^n] : \psi(x, n)\}$ is a set. It is now easy to see that $\{x \in F[A^n] : \psi(x, n)\} = [A]^n$ is a set, since by its construction, $[A]^n$ is a subset of A that has bijection with n (the property there is a function from n to y does not restrict more subsets than what we need).

(5) Let $F(x) = [A]^x$ where $[A]^x$ is a class of all subsets of A that has a bijection with x. It is routine to check that it is a class function. Since $[A]^n$ is a set for each $n \in \mathbb{N}$ by (4) and $\mathbb{N}$ is a set, then $F[\mathbb{N}] = \{[A]^n : n \in \mathbb{N}\}$ is also a set by replacement axiom. Since $\{[A]^n : n \in \mathbb{N}\}$ is a set, then by union axiom, $[A]^{<\omega} = \bigcup\{[A]^n : n \in \mathbb{N}\}$ is also a set.