Training Exam

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Test Exam, distributed 22 May 2013

Name:

UvA Student ID:

General comments.

1. **The time for this exam is 3 hours (180 minutes).**
2. **There are 90 points in the exam: 45 points are sufficient for passing.**
3. **Please mark the answers to the questions in Exercise I on this sheet by crosses.**
4. **Make sure that you have your name and student ID on each of the sheets you are handing in.**
5. **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.**
6. **No talking during the exam.**
7. **Cell phones must be switched off and stowed.**

Exercise I	
Exercise II	
Exercise III	
Exercise IV	
Exercise V	
Total	

The axioms of set theory

Some of the axioms use abbreviations such as $x \subseteq y$ for $\forall z(z \in x \to z \in y)$. Abbreviations referring to uniquely defined objects, such as $\varnothing$, $\{x\}$, or $x \cup y$ require a sufficient base theory to prove that the objects exist and are unique. These axioms are to be understood in the presence of this base theory. $\mathrm{Funct}(f)$ is the abbreviation for "f is a function".

(Ex) $\exists x(x = x)$ (Existence axiom).

(Empty) $\exists x \forall y(y \notin x)$ (Empty set axiom).

(Ext) $\forall x \forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$ (Axiom of extensionality).

(Aus) Let φ be a formula of the language of set theory (possibly with free parameters). $\forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z))$ (Axiom scheme of separation or *Aussonderung*).

(Pair) $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$ (Axiom of pairing).

(Pow) $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$ (Power set axiom).

(BinUn) $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w \in x \vee w \in y))$ (Axiom of binary union).

(Un) $\forall x \exists y \forall z(z \in y \leftrightarrow \exists w(w \in x \wedge z \in w))$ (Union axiom).

(Inf) $\exists x(\varnothing \in x \wedge \forall z(z \in x \to z \cup \{z\} \in x))$ (Axiom of infinity).

(Repl) Let φ be a formula of the language of set theory (possibly with free parameters) in two free variables. $\big((\forall x \exists y \varphi(x, y)) \wedge (\forall x \forall y \forall z(\varphi(x, y) \wedge \varphi(x, z) \to y = z))\big) \to \forall x \exists y \forall z(\exists w(w \in x \wedge \varphi(w, z)) \to z \in y)$ (Axiom scheme of replacement).

(Found) $\forall x(x \neq \varnothing \to \exists y(y \in x \wedge y \cap x = \varnothing)$ (Axiom of foundation).

(Found*) Let φ be a formula of the language of set theory (possibly with free parameters). $\exists x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y(y \in x \to \neg\varphi(y)))$ (Axiom scheme of foundation).

AC $\forall x(\forall y(y \in x \to y \neq \varnothing) \to \exists f(\mathrm{Funct}(f) \wedge \mathrm{dom}(f) = x \wedge \forall y(y \in x \to f(y) \in y))$ (Axiom of choice).

FST: (Ext) + (Aus) + (Pair) + (Pow) + (Un) (Finite set theory).

Z: FST + (Inf) (Zermelo set theory).

ZF$^-$: Z + (Repl) (Zermelo-Fraenkel set theory without foundation).

ZF: ZF$^-$ + (Found) (Zermelo-Fraenkel set theory).

ZFC: ZF + AC (Zermelo-Fraenkel set theory with choice).

Exercise I (30 points).

Find the correct answer (2 points each). Each of the questions has only one correct answer. Please pay attention to negations and whether the question asks for "false" or "true". No points are subtracted for wrong answers.

1. One of the following ordinal equalities is false. Which one?

   ☐ **A** $2 \cdot \omega = 3 \cdot \omega$.

   ☐ **B** $3 + \omega + \omega^2 = \omega + 3 + \omega^2$.

   ☐ **C** $\omega^2 + 3 = 3 + \omega^2$.

   ☐ **D** $12 \cdot (5 + \omega) = 60 \cdot \omega$.

2. Suppose that λ is a limit cardinal. Which of the following statements is provable in ZFC?

   ☐ **A** $\mathrm{cf}\lambda = \aleph_0$.

   ☐ **B** The cardinal λ is regular.

   ☐ **C** $\lambda = \aleph_\lambda$.

   ☐ **D** None of the above.

3. In ZFC, one of the following statements is equivalent to $2^{\aleph_0} = \aleph_1$. Which one?

   ☐ **A** There is a bijection between the power set of $\mathbb{N}$ and the real numbers.

   ☐ **B** Every uncountable set of real numbers contains a set of cardinality $2^{\aleph_0}$.

   ☐ **C** Every set that is equinumerous to the real numbers is uncountable.

   ☐ **D** Every uncountable set has cardinality $2^{\aleph_0}$.

4. Let κ, λ, and μ be cardinals. One of the following statements is not generally true. Which one?

   ☐ **A** $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

   ☐ **B** $\kappa \leq \lambda$ implies $\kappa^\mu \leq \lambda^\mu$.

   ☐ **C** $\kappa \leq \lambda$ implies $\mu^\kappa \leq \mu^\lambda$.

   ☐ **D** $\kappa < \lambda$ implies $\mu^\kappa < \mu^\lambda$.

5. Consider the structure **HF** of hereditarily finite sets (i.e., $\mathbf{V}_\omega$ in the von Neumann hierarchy). Which of the following axioms does not hold in this structure?

   ☐ A (Repl),

   ☐ B AC,

   ☐ C (Inf),

   ☐ D (Pair).

6. One of the following cardinal numbers is *not* singular, which one?

☐ A $\aleph_1$.

☐ B $\aleph_{\aleph_1}$.

☐ C $\aleph_{\aleph_{\aleph_1}}$.

☐ D $\aleph_{\aleph_{\aleph_{\aleph_1}}}$.

7. Let X and Y be sets of ordinals. One of the following sets is **not** in general an ordinal. Which one?

☐ A $\bigcap X$,

☐ B $\bigcup Y$,

☐ C $\bigcup(X \cup Y)$,

☐ D $X \cup Y$.

8. Let $\kappa := \aleph_7$. In the following, $+$ and $\cdot$ denote the ordinal operations. One of the equations is **false**. Which one?

☐ A $|\kappa \cdot 8| = \kappa$,

☐ B $|\kappa + 8| = \kappa$,

☐ C $|\kappa \cdot \aleph_8| = \aleph_8$,

☐ D $|8 + \kappa| = \aleph_8$.

9. The statement "there are no cardinal numbers between $\aleph_0$ and $\aleph_1$" is...

☐ A ...provable in ZF,

☐ B ...provable in ZFC, but not in ZF,

☐ C ...equivalent to the Continuum Hypothesis in the base theory ZFC.

☐ D None of the above.

10. An ordinal γ was called a *gamma number* if for all $\alpha, \beta \in \gamma$, we have that $\alpha + \beta \in \gamma$. Only one of the following ordinals is a gamma number. Which one?

☐ A $\omega + \omega$,

☐ B $\omega + \omega^2$,

☐ C $\omega^2 + \omega$,

☐ D $\omega^2 + \omega^2$.

11. Which of the following is *not* provable in ZF? (Here, $\wp$ denotes the power set symbol.)

☐ A There is a surjection $f : \mathbb{R} \to \omega$.

☐ B There is a surjection $f : \mathbb{R} \to \omega_1$.

☐ C There is a surjection $f : \mathbb{R} \to \omega_2$.

☐ D There is an injection $f : \omega_1 \to \wp(\mathbb{R})$.

12. Which of the following statements is provable in ZFC?

☐ A There is a limit cardinal of cofinality $\aleph_\omega$.

☐ B There is a limit cardinal of cofinality $\aleph_4$.

☐ C There is an uncountable successor cardinal of cofinality $\aleph_0$.

☐ D There is a regular limit cardinal of cofinality $\aleph_3$.

13. An ordinal operation F is a class assigning an ordinal $F(\alpha)$ to each ordinal α. It is called *monotone* if for all $\alpha < \beta$, we have $F(\alpha) < F(\beta)$. It is called *continuous* if for limit ordinals λ, we have $F(\lambda) = \bigcup\{F(\alpha)\,;\, \alpha < \lambda\}$. An ordinal γ is called a *fixed point of* F if $F(\gamma) = \gamma$. Only one of the following statements is **not** provable for a monotone and continuous ordinal operation. Which one?

☐ A There is a fixed point $\gamma \geq \aleph_{\aleph_1}$.

☐ B There is a fixed point of cofinality $\aleph_8$.

☐ C There are infinitely many fixed points.

☐ D There is a regular fixed point.

14. Work in ZF^- as a base theory. One of the following statements is true. Which one?

☐ A The axiom scheme of foundation is equivalent to the axiom of foundation.

☐ B The axiom scheme of foundation is equivalent to the axiom of foundation plus the axiom of choice.

☐ C The axiom scheme of foundation is provable.

☐ D The axiom scheme of replacement implies the axiom scheme of foundation.

15. Consider the integers $\mathbb{Z}$ with their natural order $<$ and their natural multiplication $\cdot$. One of the following sets is wellordered by $<$. Which one?

☐ A $\mathbb{Z}\backslash\{0\}$,

☐ B $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,

☐ C $\{z \in \mathbb{Z}\,;\, z < 0 \wedge \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,

☐ D $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = x \cdot x)\}$.

Exercise II (9 points).

Work in the base theory (Ext) + (Aus) + (Pow) + (Un) (i.e., FST without pairing). Prove that the axiom scheme of replacement implies the axiom of pairing.

Exercise III (15 points).

Work in the base theory ZF. The *wellordering theorem* states that for every set X, there is a relation $R \subseteq X \times X$ such that (X, R) is a wellorder; we abbreviate this statement by WO. Prove that AC implies WO.

Exercise IV (16 points).

If X is any set, we denoted by $\aleph(X)$ the smallest ordinal α such that there is no injection from α into X, called the *Hartogs aleph of* X. We proved in ZF that the Hartogs aleph exists. Prove in ZF that the following two statements are equivalent:

1. X is wellorderable (i.e., there is a relation $R \subseteq X \times X$ such that (X, R) is a wellorder), and
2. there is an injection $i : X \to \aleph(X)$.

Exercise V (20 points).

Work in ZF.

1. Show that every countable strict linear order (L, R) can be embedded order-preservingly into the rational numbers with their natural order $(\mathbb{Q}, <)$ (i.e., there is an order-preserving injection $f : L \to \mathbb{Q}$).
2. Give an example of a strict linear order (L, R) that cannot be embedded order-preservingly into the rational numbers.
3. Is there an uncountable ordinal that can be embedded order-preservingly into the real numbers with their natural order? Give a proof for your answer. (**Hint.** Use some of the basic properties of the metric or topological structure of the real numbers.)