Homework Set #7

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 5 April 2013, 1pm**.

1. Remember the *cardinal* and the *ordinal* definitions of addition and multiplication on the natural numbers: if n and m are natural numbers, we defined $n+m$ as the unique natural number k such that there are $X \subseteq k$ and $Y \subseteq k$ with $X \cap Y = \varnothing$, $X \cup Y = k$, $n \sim X$, and $m \sim Y$; we defined $n \cdot m$ as the unique natural number k such that $n \times m \sim k$; furthermore, we defined functions add_n and mult_n by recursion via $\mathrm{add}_n(0) = n$, $\mathrm{add}_n(S(m)) = S(\mathrm{add}_n(m))$, $\mathrm{mult}_n(0) = 0$, and $\mathrm{mult}_n(S(m)) = \mathrm{add}_n(\mathrm{mult}_n(m))$.

   Prove that for all n and m, we have

   $$n + m = \mathrm{add}_n(m) \text{ and}$$
   $$n \cdot m = \mathrm{mult}_n(m).$$

2. We call a set X *union-created* if there is an $A \subseteq X$ such that $X = \bigcup A$. If X is union-created, we say that X has the *uniqueness property* if

   $$\forall A \subseteq X \left(\bigcup A = A \rightarrow (A = X \vee A = \varnothing) \right).$$

   (a) Show that if α is a union-created ordinal, then it has the uniqueness property if and only if $\alpha = \mathbb{N}$ or $\alpha = 0$.

   (b) Give an example of an ordinal α that is union-created and does not have the uniqueness property.

3. Let (X, R) be a strict partial order, i.e., R is irreflexive and transitive. As before, for $x \in X$, we let $\mathrm{IS}(x) := \{x' \in X\,;\, x'Rx\}$. We said that $A \subseteq X$ is called an *initial segment* if for all $a \in A$ and xRa, we have $x \in A$. We call an initial segment *proper* if $A \neq X$. In class, we proved that if (X, R) is a wellorder (i.e., R is in addition wellfounded and linear), then every proper initial segment is of the form $\mathrm{IS}(x)$ for some x. Show that the two additional properties are necessary, i.e.,

   (a) find a strict linear order (i.e., irreflexive, transitive and linear) that refutes the claim, and

   (b) find a wellfounded strict partial order that refutes the claim.

4. Prove the two statements mentioned in class:

   (a) If α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal.

   (b) If A is a set of ordinals, then $\bigcup A$ is an ordinal.

5. Prove in ZF^- that for every set X there is a unique set $\text{tcl}(X)$ (for "transitive closure") such that

   - $X \subseteq \text{tcl}(X)$,
   - $\text{tcl}(X)$ is transitive, and
   - for any transitive set $T \supseteq X$, we have $\text{tcl}(X) \subseteq T$.

6. A set X is called *hereditarily finite* if X is finite and every $x \in \text{tcl}(X)$ is finite. Show that the following are equivalent:

   (a) X is hereditarily finite, and

   (b) $\text{tcl}(X)$ is finite.

⋆ *Extra question for extra credit.* The Recursion Theorem allows us to do the model constructions (previously done informally in the first month of the class) inside models of ZF^-. Prove the following claims in ZF^-:

   (a) There is a set H of all hereditarily finite sets; moreover, $(H, \in)$ satisfies all of the axioms of FST.

   (b) There is a set X such that $(X \ \in)$ satisfies all of the axioms of Z. Find an instance of the Axiom Scheme of Replacement that fails in $(X, \in)$ and use this to show that Z does not imply the Axiom Scheme of Replacement.

   (**Hint.** Do the iterated HF-construction starting from the set H obtained in the first part of this exercise.)