Homework Set #5

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

The homework can be handed it electronically to Takanori Hida (`t.hida@uva.nl`) or on paper in the Friday lectures. Please hand in the homework before the start of the Friday lecture (1pm). Late homework will not be accepted.

The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Friday 15 March 2013, 1pm**.

1. Let X and Y be two sets and let $\leq$ be a binary relation on Y. Consider a function $f : X \to Y$ and define
$$x \leq^* x' :\iff f(x) \leq f(x').$$
We call $\leq^*$ the *pullback* of $\leq$. As usual, we write $y < y'$ for $y \leq y' \wedge y \neq y'$ and, similarly, $x <^* x'$ for $x \leq^* x' \wedge x \neq x'$. Check whether the following properties are "pulled back", i.e., whether the statement "if $\leq$ has property **X**, then $\leq^*$ has property **X**" is true. If it is true, give a proof; if not, give a counterexample:

   (a) linearity: $\forall y \forall y' (y \leq y' \vee y' \leq y)$;

   (b) reflexivity: $\forall x (x \leq x)$;

   (c) transitivity: $\forall x \forall x' \forall x'' (x \leq x' \wedge x' \leq x'' \to x \leq x'')$;

   (d) anti-symmetry: $\forall y \forall y' (y \leq y' \wedge y' \leq y \to y = y')$;

   (e) well-foundedness: $\forall A \subseteq Y (A \neq \varnothing \to \exists a (a \in A \wedge \forall y (y < a \to x \notin A)))$.

2. Use the set-up of Exercise 1. and assume in addition that the function f is injective. We call a structure $(Y, \leq)$ a *(non-strict) wellorder* if it is a well-founded linear order (i.e., well-founded, linear, reflexive, transitive, and anti-symmetric). Show that if $(Y, \leq)$ is a well-order, then $(X, \leq^*)$ is also a well-order. (Of course, you can use the positive answers you obtained in Exercise 1.).

3. Let $(X, \leq)$ be a linear order. As before, we abbreviate $x \leq x' \wedge x \neq x'$ by $x < x'$. As in class, we call a function $f : \mathbb{N} \to X$ an *infinite descending sequence* if for all n, we have that $f(n+1) < f(n)$. Similarly, we call a function $f : \mathbb{N} \to X$ an *infinite ascending sequence* if for all n, we have that $f(n) < f(n+1)$. If f is an infinite ascending or descending sequence and $x \in X$, we say that x *lies below* f if for all $n \in \mathbb{N}$, $x < f(n)$. We say that $x \in X$ is a *minimal element* if for all $x' \in X$, we have $x \leq x'$.

   Give examples of linear orders with the following properties:

   (a) X has an infinite ascending sequence f, an element a that is an upper bound of $\mathrm{ran}(f)$ and another infinite ascending sequence g such that all elements of $\mathrm{ran}(g)$ lie above a.

   (b) X has a minimal element, an infinite descending sequence f, an element a that is an upper bound of $\mathrm{ran}(f)$, and both infinite ascending and descending sequences g and h such that a lies below both g and h.