Homework Set #12

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Please hand in the homework before the start of the Wednesday *werkcollege* (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Wednesday 15 May 2013, 1pm**.

Please keep track of the time that you spend on this homework and write it on top of your answer sheet. Of course, this information will not be used in a personalized way, but only to get an estimate for the average weekly work load.

1. Let X be a set of nonempty sets and Y an arbitrary set. Let $P := \{f\,;\,\mathrm{dom}(f) \subseteq X$ and f is a choice function for $\mathrm{dom}(X)\}$ and $Q := \{f\,;\,\mathrm{dom}(f) = A \subseteq Y, \mathrm{ran}(f) = A \times A$ and f is an injection$\}$. Order the elements of both P and Q by inclusion, i.e.,

$$f \leq g :\iff \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \text{ and } g{\upharpoonright}\mathrm{dom}(f) = f.$$

   Show that both $(P, \leq)$ and $(Q, \leq)$ are chain-complete, i.e., every chain has an upper bound.

2. In class, we showed that if α is a limit ordinal, then $(\{0\} \times \alpha) \cup (\{1\} \times \alpha) = 2 \times \alpha \sim \alpha$. Generalize this to successor ordinals α.

3. Let $L := \{R \subseteq \mathbb{N} \times \mathbb{N}\,;\,(\mathbb{N}, R)$ is a linear order$\}$. Define an equivalence relation $\equiv$ by $R \equiv R' :\iff (\mathbb{N}, R)$ and $(\mathbb{N}, R')$ are isomorphic as linear orders (i.e., there is a bijection that preserves the order in both directions). Show that there is an injection from $\wp(\mathbb{N})$ into $L/{\equiv}$.

   (**Hint.** As a warm-up, consider the following orders: L_1 consists of a copy of the integers, $\mathbb{Z}$, then a single point, and then another copy of the integers; L_2 consists of a copy of the integers, $\mathbb{Z}$, then two points, and then another copy of the integers (see below for an informal graphical representation). Show that L_1 and L_2 are not isomorphic. Generalize this idea to assign different linear orders to each subset $X \subseteq \mathbb{N}$.)

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots < \mathsf{p} < \cdots < -2 < -1 < 0 < 1 < 2 < \cdots \qquad (L_1)$$

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots < \mathsf{p}_0 < \mathsf{p}_1 < \cdots < -2 < -1 < 0 < 1 < 2 < \cdots \qquad (L_2)$$

Extra credit exercise (for students with a firm background in topology). Tychonoff's theorem (in topology) says that arbitrary products of compact topological spaces are compact (note that we do not require the spaces to be Hausdorff). This theorem is usually proved in a topology class using the Axiom of Choice. Show in ZF that Tychonoff's theorem implies AC.

(**Hint.** If X is a family of nonempty sets, $x \in X$ and $\mathsf{p} \notin \bigcup X$, define $S_x := x \cup \{\mathsf{p}\}$ and $\tau_x := \{S_x, x, \{\mathsf{p}\}, \varnothing\}$. Note that for each x, (S_x, τ_x) is a compact space. Consider $\prod_{x\in X}(S_x, \tau_x)$ and find a family of open sets that does not cover the product in such a way that an element that is not covered by the family will be a choice function for X.)