Homework Set #10

Axiomatische Verzamelingentheorie
2012/13: 2nd Semester
Universiteit van Amsterdam

Please hand in the homework before the start of the Wednesday *werkcollege* (1pm). Late homework will not be accepted. **Takanori and Zhenhao do not accept electronic submissions anymore**. The homework should contain the full names and student ID numbers of all students who contributed. Each homework solution should have at most two names of students. The homework handed in must be the work of the students named on that homework solution. If your homework is handwritten, make sure that it is legible. Also, make sure that you write in complete English sentences.

This homework set is due on **Wednesday 24 April 2013, 1pm**.

Please keep track of the time that you spend on this homework and write it on top of your answer sheet. Of course, this information will not be used in a personalized way, but only to get an estimate for the average weekly work load.

1. Prove the Cantor-Schröder-Bernstein theorem by filling in the details in the following proof sketch:

   (a) Assume that there are injections $f : A \to B$ and $g : B \to A$.

   (b) Define $A_0 := A \backslash \mathrm{ran}(g)$ and $A + n + 1 := \{g(f(x))\,;\, x \in A_n\}$. Let $A^* := \bigcup\{A_n\,;\, n \in \mathbb{N}\}$. Note that $A \backslash A^* \subseteq \mathrm{ran}(g)$.

   (c) Define
   $$h : x \mapsto \begin{cases} f(x) & \text{if } x \in A^*\text{, and} \\ g^{-1}(x) & \text{if } x \notin A^*. \end{cases}$$
   Show that h is defined for all $x \in A$.

   (d) Prove that h is a bijection between A and B.

   (e) Conclude that the Cantor-Schröder-Bernstein theorem holds.

2. In class, we said that an ordinal α is called an *infinite cardinal* if there is an ordinal γ such that $\alpha = \aleph_\gamma$. We also mentioned that an ordinal α is called *initial* if there is no $\beta < \alpha$ such that $\alpha \preceq \beta$. Let $\alpha \geq \omega$. Show that the following are equivalent:

   (a) α is an infinite cardinal, and

   (b) α is an initial ordinal.

3. Define the following ordinal operation:
   $$\begin{aligned} \alpha_0 &:= 0, \\ \alpha_{\gamma+1} &:= \aleph(\alpha_\gamma), \\ \alpha_\lambda &:= \bigcup\{\alpha_\gamma\,;\, \gamma < \lambda\} \text{ (for limit ordinals } \lambda). \end{aligned}$$
   Show that there is a β such that for all $\gamma \geq \beta$, we have $\alpha_\gamma = \aleph_\gamma$. Compute the least such β.