Manipulation in Approval Voting

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- Research at the interface of
  - mathematical economics: social choice, game theory, decision theory
  - computer science and AI, multiagent systems, logic

- Some examples:
  - voting: computational hardness as a barrier against manipulation
  - preference representation in combinatorial domains
  - logic-based modelling of social choice procedures
  - multiagent resource allocation and fair division

- Activities at the ILLC:
  - MoL Course on Computational Social Choice (in Spring)
  - Computational Social Choice Seminar
Vote Manipulation

Suppose the *plurality rule* (as in most real-world situations) is used to decide the outcome of an election: the candidate receiving the highest number of votes wins.

Assume the preferences of the people in, say, Florida are as follows:

- 49%: Bush $\succ$ Gore $\succ$ Nader
- 20%: Gore $\succ$ Nader $\succ$ Bush
- 20%: Gore $\succ$ Bush $\succ$ Nader
- 11%: Nader $\succ$ Gore $\succ$ Bush

So even if nobody is cheating, Bush will win in a plurality contest.

**Issue**: In a pairwise competition, Gore would have defeated anyone.

**Issue II**: It would have been in the interest of the Nader supporters to *manipulate*, i.e. to misrepresent their preferences.
Talk Outline

- Background: the Gibbard-Satterthwaite Theorem
- Background: Approval Voting
- Tie-Breaking and Preferences over Sets of Candidates
- Results: Manipulation in Approval Voting
- Conclusion
The Gibbard-Satterthwaite Theorem

Theorem 1 (Gibbard-Satterthwaite) Every voting rule for three or more candidates must be either dictatorial or manipulable.

Let $C$ be a finite set of candidates and let $\mathcal{P}$ the set of all linear orders over $C$. A voting rule for $n$ voters is a function $f : \mathcal{P}^n \to C$, selecting a single winner given the (reported) voter preferences.

A voting rule is dictatorial if the winner is always the top candidate of a particular voter (the dictator).

A voting rule is manipulable if there are situations where a (single) voter can force a preferred outcome by misreporting his preferences.


The Gibbard-Satterthwaite Theorem

Theorem 1 (Gibbard-Satterthwaite)  *Every voting rule for three or more candidates must be either dictatorial or manipulable.*

Despite its generality, the Gibbard-Satterthwaite Theorem may not apply in all cases (at least not immediately):

- The theorem presupposes that a ballot is a full preference ordering over all candidates. Plurality voting, for instance, does not satisfy this condition (although it’s manipulable anyway).

- The theorem also presupposes that there is a *unique* way of casting a *sincere ballot* for any given preference ordering.

We will concentrate on the second “loophole”. We can imagine various situations in which there may be more than one way of casting a sincere vote . . .
Approval Voting

In approval voting, a ballot is a subset of the set of candidates. These are the candidates the voter approves of. The candidate receiving the most approvals wins (we’ll discuss tie-breaking later).

Approval voting has been used by several professional societies, such as the American Mathematical Society (AMS).

We assume each voter has a preference ordering $\preceq$ over candidates (which is antisymmetric, transitive and total).

A given voter’s ballot is called sincere if all approved candidates are ranked above all disapproved candidates according to that voter’s $\preceq$.

Example: If $A \succ B \succ C$, then $\{A\}$, $\{A, B\}$ and $\{A, B, C\}$ are all sincere ballots. The latter has the same effect as abstaining.

Tie-breaking and Preferences over Sets of Candidates

We call the candidates with the most approvals the *pre-winners*. If there are two or more pre-winners, we have to use a suitable *tie-breaking rule* to choose a winner.

Tie-breaking is outside the control of voters (in general). So when considering to manipulate, they have to do so in view of their preferences over sets of pre-winners.

- Given a voter’s preferences $\leq$ over *individual candidates*, what can we say about his preferences $\sqsubseteq$ over *sets of pre-winners*?

- Answer: *depends* (we’ll consider several possible axiomatisations)
Axioms for Preferences over Sets of Candidates

We can try to axiomatise the range of possible choices for \( \subseteq \) we wish to admit. One very reasonable option is this:

- \( \subseteq \) is reflexive and transitive.
- (DOM) \( A \subseteq B \) if \( \#A = \#B \) and there exists a surjective mapping \( f : A \to B \) such that \( a \preceq f(a) \) for all \( a \in A \).
- (ADD) \( A \subseteq B \) if \( A \subset B \) and \( a \preceq b \) for all \( a \in A \) and all \( b \in B \setminus A \).
- (REM) \( A \subseteq B \) if \( B \subset A \) and \( a \preceq b \) for all \( a \in A \setminus B \) and all \( b \in B \).
**An Example for Successful Manipulation**

Suppose all but one voter have voted. This final voter wants to manipulate. His preferences are: $4 \succ 3 \succ 2 \succ 1$.

Suppose 3 and 1 each got 10 votes so far (*pivotal* candidates); 4 and 2 each got 9 (*subpivotal* candidates). The final voter can

- force outcome $431$ by voting $[4]$;
- force outcome $4321$ by voting $[42]$;
- force outcome $1$ by voting $[421]$, $[41]$, $[21]$ or $[1]$; or

Outcomes $431$, $3$ and $4321$ are *undominated* according to our axioms. If (and only if) the final voter prefers $4321$ amongst these, he has an incentive to submit the insincere ballot $[42]$. 
The Case of Three Candidates

Recall that the Gibbard-Satterthwaite hits once we move from two to three candidates. The previous example shows that approval voting is certainly manipulable in the case of four candidates . . .

**Theorem 2 (Three candidates)** *In approval voting with three candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot.*

This is a special case of a result by Brams and Fishburn (1978).

Proof of Theorem 2

Check all possible cases. For each candidate, distinguish whether she is pivotal (P), subpivotal (S) or insignificant (I). At least one has to be pivotal, so there are $3^3 - 2^3 = 19$ possible situations.

?- table(3).

| | [100] | [110] | [111] | [001] | [010] | [011] | [101] |
|-------------------------------|
| ... 9 obvious cases of the form P__ omitted |
| SPP | 321 | 2 | 21 | 1 | 2 | 21 | 1 |
| SPS | 32 | 2 | 2 | 21 | 2 | 2 | 321 |
| SPI | 32 | 2 | 2 | 2 | 2 | 2 | 32 |
| SSP | 31 | 321 | 1 | 1 | 21 | 1 | 1 |
| SIP | 31 | 31 | 1 | 1 | 1 | 1 | 1 |
| IPP | 21 | 2 | 21 | 1 | 2 | 21 | 1 |
| IPS | 2 | 2 | 2 | 21 | 2 | 2 | 21 |
| IPI | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| ISP | 1 | 21 | 1 | 1 | 21 | 1 | 1 |
| IIP | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

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The Case of Four Candidates

We know that manipulation is possible with four candidates (see earlier example). But how many problematic situations are there?

Answer: Just one!

Theorem 3 (Four candidates) In approval voting with four candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot, unless he strictly prefers 4321 over both 431 and 3.

The proof has been derived automatically using a computer program that checks all possible scenarios.
The Case of Five Candidates

?- theorem(5).

Theorem: In approval voting with 5 candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot, unless his preferences over sets of candidates satisfy one of the following 10 conditions:

-- 54321 strictly dominates all of 5431, 4.
-- 54321 strictly dominates all of 5421, 4.
-- 54321 strictly dominates all of 542, 4.
-- 5432 strictly dominates all of 542, 4.
-- 54321 strictly dominates all of 531, 5431, 3.
-- 5321 strictly dominates all of 531, 3.
-- 4321 strictly dominates all of 431, 3.
Changing the Axioms

Next we assume that voters are *expected-utility maximisers*, but we want to drop any assumptions about the tie-breaking rule used.

Fully *general tie-breaking* can be axiomatised like this:

- (GEN) $A \succeq B$ if $a \preceq b$ for all $a \in A$ and all $b \in B$.

**Question:** How does this affect our manipulability results?
The Case of General Tie-Braking

By widening the range of conceivable orderings $\preceq$ over pre-winners, we have to give up our sincerity theorem for three candidates:

?- theorem(3). % based on (GEN) only

Theorem: In approval voting with 3 candidates, suppose that all but one voter have cast their ballot. Then the final voter has no incentive to cast an insincere ballot, unless his preferences over sets of candidates satisfy one of the following 2 conditions:

-- 321 strictly dominates all of 32.
-- 21 strictly dominates all of 31, 321.

For four candidates, there are already 19 such exceptions ...
The Case of Uniform Tie-Breaking

If we are more specific about how ties are broken and how voters form preferences over sets of pre-winners we can obtain stronger results:

**Theorem 4 (Expected-utility maximisers)** In approval voting with uniform tie-breaking, suppose that all but one voter have cast their ballot. Then, if the final voter is an expected-utility maximiser, he has no incentive to cast an insincere ballot.

Remark: We need not make any assumptions regarding the actual utility functions used by the voters, other than that they are compatible with their preference orderings \(\preceq\).
Discussion

What does this last result tell us about the merits of approval voting?

- Uniform tie-breaking in combination with the assumption that voters are expected-utility maximisers is arguably the most relevant scenario in practice. \( \sim \) no need to be *insincere*

- But *manipulation* is still possible! Voters can strategise by choosing the most promising of their sincere ballots. We only show that there is no need for them to consider insincere ballots.

- The number of sincere ballots is *linear*; the number of insincere ballots is *exponential* in the number if candidates.
Conclusion

- **Basic idea:** The presence of *multiple sincere ballots* may allow us to circumvent the Gibbard-Satterthwaite Theorem in the sense that some sincere ballot may always be optimal.

- **Results:** For *approval voting*, it turns out that this is indeed the case for several interesting scenarios:
  - If all voters are *optimistic* or *pessimistic* (*not shown*).
  - If there are at most *three* candidates and one of these hold:
    * Voter preferences are governed by our “*reasonable axioms*”.
    * A *rational chair* is breaking the ties (*not shown*).
  - If *uniform tie-breaking* is used and the voters are *expected-utility maximisers*.