Provability Logics of Constructive Theories

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Core Logic,
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Overview

Provability Logic

Friedman’s Classical Problem

Friedman’s Problem: the Constructive Variant
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The First Incompleteness Theorem

Let $T$ be a theory that interprets a reasonable weak theory of arithmetic like Buss’ $S^1_2$. In this talk we will also consider the possibility that such a theory is constructive.

We write $\square_T A$ for $\text{Prov}_T(\lceil A \rceil)$.

The Gödel sentence for $T$:

$\quad T \vdash G \iff \neg \square_T G$.

We have:

$T \vdash G \Rightarrow T \vdash \square_T G$

$\Rightarrow T \vdash \neg G$

$\Rightarrow T \vdash \bot$
The Second Incompleteness Theorem

We formalize the above reasoning in $T$.

$$T \vdash \Box_T G \rightarrow \Box_T \Box_T G$$

$$\rightarrow \Box_T \neg G$$

$$\rightarrow \Box_T \bot$$

We find $T \vdash G \leftrightarrow \neg \Box_T \bot$.

So the second incompleteness theorem follows from the first.
We interpret the language of modal propositional logic into $T$ via interpretations $(\cdot)^*$ that send the propositional atoms to arbitrary sentences, commute with the propositional connectives and satisfy:

$$(\Box \phi)^* := \Box T \phi^*.$$ 

We say that $\phi$ is (an) arithmetically valid (scheme) for $T$ iff, for all $(\cdot)^*$, we have $T \vdash \phi^*$. 
Löb’s Logic

Löb’s Logic aka GL is the modal propositional theory axiomatized by classical propositional logic plus the following axioms and rules.

\begin{align*}
\text{L1.} & \quad \vdash (\Box \phi \land \Box(\phi \rightarrow \psi)) \rightarrow \Box \psi, \\
\text{L2.} & \quad \vdash \Box \phi \rightarrow \Box \Box \phi, \\
\text{L3.} & \quad \vdash \Box (\Box \phi \rightarrow \phi) \rightarrow \Box \phi, \\
\text{L4.} & \quad \vdash \phi \Rightarrow \vdash \Box \phi.
\end{align*}

Löb’s Logic is arithmetically sound for all classical theories that interpret Buss’ $S_2^1$. It is arithmetically complete for all classical $\Sigma^0_1$-sound theories that interpret EA (Elementary Arithmetic) aka $I\Delta_0 + \text{Exp}$. (Solovay 1976)
Some Theorems

GL is complete for finite transitive irreflexive Kripke models.

A variable $p$ is \textit{modalized} in $\phi$ iff all its occurrences are in the scope of a box. We write $\boxdot \phi$ for $\phi \land \square \phi$.

\textit{Bernardi, de Jongh, Sambin}: Suppose $p$ is modalized in $\phi p$.

\begin{itemize}
\item $\vdash (\boxdot (p \leftrightarrow \phi p) \land \boxdot (q \leftrightarrow \phi q)) \rightarrow (p \leftrightarrow q)$.
\end{itemize}

\textit{Sambin, de Jongh}: Suppose $p$ is modalized in $\phi p \bar{q}$. Then, there is a $\psi \bar{q}$, such that:

\begin{itemize}
\item $\vdash \psi \bar{q} \leftrightarrow \phi (\psi \bar{q}) \bar{q}$.
\end{itemize}

E.g. if $\phi p$ is $\neg \square p$, then $\psi$ is $\neg \square \bot$.

\textit{Shavrukov}: GL has uniform interpolation.
The Constructive Case

Provability Logics of theories are not monotonous in these theories!

\(iGL\) is sound for extensions of \(iS^1_2\).

Principles for Heyting Arithmetic aka HA.

Leivant’s Principle \(\vdash \Box (\phi \lor \psi) \rightarrow \Box (\phi \lor \Box \psi)\).

Markov’s Rule \(\vdash \Box \neg \neg \Box \phi \rightarrow \Box \Box \phi\).

Anti-Markov’s Rule \(\vdash \Box (\neg \neg \Box \phi \rightarrow \Box \phi) \rightarrow \Box \Box \phi\).

In classical GL plus Leivant’s Principle we have:

\[
\vdash \Box (\Box \bot \lor \neg \Box \bot) \rightarrow \Box (\Box \bot \lor \Box \neg \Box \bot) \rightarrow \Box \Box \bot
\]
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The Problem

The closed fragment of provability logic is simply the logic for zero propositional variables.

Friedman’s 35th problem was to give a decision procedure for the closed fragment of the provability logic of Peano Arithmetic, PA. (Friedman 1975) It was independently solved by van Benthem, Boolos and Bernardi & Montagna.

The van Benthem-Boolos-Bernardi-Montagna result holds for $\Sigma^0_1$-sound theories that interpret $S^1_2$. 
Degrees of Falsity

Let $\omega^+ := \omega \cup \{\infty\}$. We equip $\omega^+$ with the usual ordering and define $\infty + 1 := \infty$. Note that the successor function remains injective under this extension.

We define the modal degrees of falsity as follows.

1. $\square^0 \bot := \bot$,
2. $\square^{n+1} \bot := \square^n \bot$,
3. $\square^\infty \bot := \top$.

We have:

1. $\vdash (\square^\alpha \bot \land \square^\beta \bot) \iff \square^{\min(\alpha, \beta)} \bot$.
2. $\vdash (\square^\alpha \bot \lor \square^\beta \bot) \iff \square^{\max(\alpha, \beta)} \bot$.
3. $\vdash \square (\square^\alpha \bot \rightarrow \square^\beta \bot) \iff \square^\infty \bot$, if $\alpha \leq \beta$.
4. $\vdash \square (\square^\alpha \bot \rightarrow \square^{\beta+1} \bot) \iff \square^\beta \bot$, if $\alpha < \beta$. 
The Basic Idea

Suppose $\phi$ is a Boolean combination of degrees of falsity.

\[
\vdash \Box \phi \iff \Box \bigwedge \bigvee \pm \Box \alpha \bot \\
\iff \Box \bigwedge (\bigvee \Box \beta \bot \vee \neg \bigwedge \Box \gamma \bot) \\
\iff \Box \bigwedge (\Box \delta \bot \rightarrow \Box \varepsilon \bot) \\
\iff \bigwedge \Box (\Box \delta \bot \rightarrow \Box \varepsilon \bot) \\
\iff \Box \eta \bot
\]

We now prove, by induction on $\psi$, that any $\psi$ in the closed fragment is equivalent to a Boolean combination of degrees of falsity.
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Target Theories

We can characterize the closed fragments for HA, HA + MP, HA* and PA.

Markov’s Principle MP:

\[ \vdash (\forall x (Ax \lor \neg Ax) \land \neg \neg \exists x Ax) \rightarrow \exists x Ax. \]

Open: HA + ECT₀ and MA = HA + ECT₀ + MP.

Theories of Degrees of Falsity

We write $\alpha$ for $\square^\alpha \bot$. We consider theories in the propositional language where the degrees of falsity are treated as propositional constants.

We work in a propositional language with the constants $\alpha$ without variables. The theory Basic is axiomatized by Intuitionistic Propositional Logic plus $\vdash \alpha \rightarrow \beta$, for $\alpha \leq \beta$.

We consider extensions $\Gamma$ of Basic.

- $\Gamma$ is *p-sound* if $\Gamma \vdash \alpha \rightarrow \beta$ implies $\alpha \leq \beta$.
- $\Gamma$ is *decent* if, for every $\phi$ and for every $n$ larger than all $m$ occurring in $\phi$, we have $\Gamma \vdash n \rightarrow \phi$ implies $\Gamma \vdash \phi$.
- $\alpha_\Gamma(\phi) := \max\{\alpha \mid \Gamma \vdash \alpha \rightarrow \phi\}$. 

Salient Theories of Degrees

- **Stronglöb** := Basic + \{((\alpha \rightarrow \beta) \rightarrow \beta) \mid \beta < \alpha\},
- **Stable** := Basic + \{\neg\neg\alpha \rightarrow \alpha \mid \alpha \in \omega^+\},
- **Classical** := Basic + \{\alpha \vee \neg\alpha \mid \alpha \in \omega^+\}.

1. Basic corresponds to HA.
2. Stronglöb corresponds to HA*.
3. Stable corresponds to HA + MP.
4. Classical corresponds to PA.
Suppose $\Gamma$ is a decent theory of degrees. We define the closed fragment $AL_\Gamma$ by introducing a modal operator setting

$$\square \phi :\leftrightarrow \alpha_\Gamma(\phi) + 1.$$ 

We find that $AL_\Gamma$ is a closed fragment and that its theory of degrees of falsity is $\Gamma$.

**Intuition**: the box of $AL_\Gamma$ is the strongest or most informative box for closed modal theories compatible with $\Gamma$.

We prove $AL_\Gamma \vdash \square(\square \phi \rightarrow \phi) \rightarrow \square \phi$. In case $\alpha_\Gamma(\phi) = \infty$, we are easily done. Let $n := \alpha_\Gamma(\phi)$. We have:

1. $\vdash n \rightarrow ((n + 1) \rightarrow \phi)$, since $\vdash n \rightarrow \phi$.
2. $\not\vdash (n + 1) \rightarrow ((n + 1) \rightarrow \phi)$, since $\not\vdash (n + 1) \rightarrow \phi$.

So $\alpha_\Gamma(\square \phi \rightarrow \phi) = n$. 
Theorem

The closed fragments of HA, HA*, HA + MP and PA are respectively $AL_{Basic}$, $AL_{Stronglöb}$, $AL_{Stable}$, $AL_{Classical}$.

I.o.w., we have $CF_T = AL_{TDF_T}$ for these theories. We might say: we have ‘box-elimination’ for these fragments.