UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2007/2008; 1st Semester
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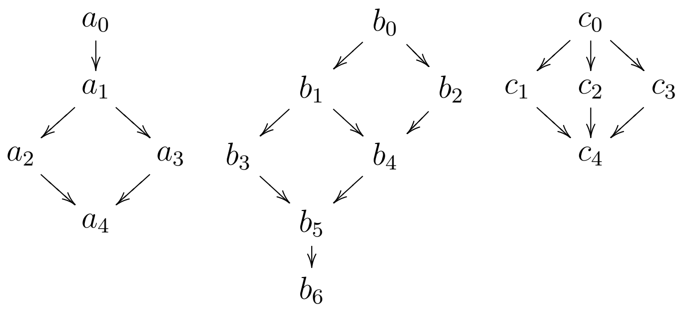
Homework Set # 11 *Deadline:* November 28th, 2007

Exercise 37 (9 points).

Consider the following three directed graphs $\mathbf{G}_0$, $\mathbf{G}_1$ and $\mathbf{G}_2$. We say that a vertex x is **below** a vertex y if there is a path from y to x. We say that x is a **bottom element** in a graph if it is below all other elements in the graph. Our three graphs have bottom elements: a_4, b_6, and c_4 are the bottom elements of $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$, respectively.



$\mathbf{G}_0$ $\mathbf{G}_1$ $\mathbf{G}_2$

In such a graph, if x and y are vertices, we define the **greatest lower bound of** x **and** y as follows. If $x = y$ or x is below y, we say that x is the greatest lower bound of x and y. Otherwise (i.e., if x and y are distinct vertices and neither is below the other), we call z the greatest lower bound of x and y if

- z is below x,
- z is below y, and
- if any w is below both x and y, then either $w = z$ or w is below z.

Notice that a greatest lower bound has to be unique if it exists. Also notice that in the three given graphs, each pair of vertices has a greatest lower bound (you do not have to prove this, but please check for yourself by trying three or four examples).

We can now define a binary operation $\wedge$ on the vertices by letting $x \wedge y$ be the greatest lower bound of x and y. Using this, we define a (possibly partial) unary operation $-$ on the vertices as follows: $-x = y$ if and only

- $x \wedge y$ is the bottom element, and
- if for any w, $x \wedge w$ is the bottom element, then either $w = y$ or w is below y.

(In other words: y is the greatest vertex such that $x \wedge y$ is the bottom element if this exists uniquely.)

(1) For each of $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$, find out whether $-$ is a total function and give an argument (1½ points each).

(2) With the given operation $-$, does $\mathbf{G}_0$ satisfy the formula $--x = x$? (Give an argument; 2 points)
(3) With the given operation $-$, does $\mathbf{G}_1$ satisfy the formula $---x = -x$? (Give an argument; 2½ points)

Exercise 38 (7 points).

(1) As mentioned in the lecture: Find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \oplus \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \oplus \mathbf{W}$ and explain why (2 points).
(2) Similarly, find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \otimes \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \otimes \mathbf{W}$ and explain why (2 points).
(3) In the first two tasks, you can choose one wellorder to be finite. Why can't both wellorders be finite in such an example (1 point)?
(4) Consider $\mathbf{L} := \langle \mathbb{Q}, \leq \rangle$ to be the rational numbers with the usual ordering. Find out whether $\mathbf{L} \oplus \mathbf{L}$ is isomorphic to $\mathbf{L}$ and give an argument (2 points).
Hint. The Cantor Isomorphism Theorem (sometimes called "back-and-forth theorem") for countable linear orders may help. If you use it, you don't have to prove it, but please state it clearly with a proper reference to the literature and make sure that you apply it properly.

Exercise 39 (6 points).

We are modelling Achilles and the turtle as a transfinite process on the real line $\mathbb{R}$. Please give arguments for all answers.

(1) Achilles' position at time t is given by A_t, the turtle's position is given by T_t. We start with $A_0 := 0$ and $T_0 := 1$. For every index i, we define $A_{i+1} := A_i + |T_i - A_i|$, $T_{i+1} := T_i + \frac{1}{2} \cdot |T_i - A_i|$, and
$$T_\infty := \lim_{i \in \mathbb{N}} T_i,$$
$$A_\infty := \lim_{i \in \mathbb{N}} A_i,$$
$$T_{\infty+\infty} := \lim_{i \in \mathbb{N}} T_{\infty+i}, \text{ and}$$
$$A_{\infty+\infty} := \lim_{i \in \mathbb{N}} A_{\infty+i}.$$
Determine the least index i such that $A_i = T_i$ (1 point). Where is Achilles at time $\infty + \infty$ (1 point)?
(2) Now the positions are given by A^*_t and T^*_t defined as follows. For each index $i \in \{0, 1, 2, ..., \infty, \infty+1, \infty+2, \infty+3, ...\}$, we define the *value* $\mathrm{v}(i)$ as follows:
$$\mathrm{v}(i) := n \text{ if } i = n \text{ or } i = \infty + n.$$
We start with $A^*_0 := 0$ and $T^*_0 := 1$. For every index i, we define $A^*_{i+1} := A^*_i + \frac{1}{2^{\mathrm{v}(i)}}$, $T^*_{i+1} := T^*_i + \frac{1}{2^{\mathrm{v}(i)+1}}$, and
$$T^*_\infty := \lim_{i \in \mathbb{N}} T^*_i,$$
$$A^*_\infty := \lim_{i \in \mathbb{N}} A^*_i,$$
$$T^*_{\infty+\infty} := \lim_{i \in \mathbb{N}} T^*_{\infty+i}, \text{ and}$$
$$A^*_{\infty+\infty} := \lim_{i \in \mathbb{N}} A^*_{\infty+i}.$$
Compute $A^*_{\infty+5}$, $T^*_{\infty+12}$, $A^*_{\infty+\infty}$ and $T^*_{\infty+\infty}$ (1 point each).