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A NEW PROOF OF SAHLQVIST'S THEOREM ON MODAL DEFINABILITY AND COMPLETENESS

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There are not many global results on modal logics. One of these is the following theorem by Sahlqvist on completeness and correspondence for a wide class of modal formulae (including many well known logics, e.g. D, T, B, $S4$, $K4$, $S5$,...) (see [S]).

SAHLQVIST'S THEOREM. *Let A be any modal formula which is equivalent to a conjunction of formulae of the form $\Box^m(A_1 \to A_2)$, where $m \geq 0$, A_2 is positive and A_1 is obtained from propositional variables and their negations applying $\wedge$, $\vee$, $\Diamond$, and $\Box$ in such a way that no positive occurrence of a variable is in a subformula of the form $B_1 \vee B_2$ or $\Diamond B_1$ within the scope of some $\Box$. Then A corresponds effectively to a first order formula, and $L + A$ is canonical whenever L is a canonical logic.*

A formula A satisfying the above conditions is henceforth called a *Sahlqvist formula*. Unfortunately, till now, the only complete proof was the original proof of Sahlqvist (a proof of the correspondence half has also been given by van Benthem [vB]). It is so complicated and long that even in an advanced textbook of modal logic [HC] it has not found a place. Here, by considering general frames as topological spaces, an attitude which we developed in [TD], we give a proof of Sahlqvist's theorem simplified to such an extent that one can easily grasp the key idea on which it is based and apply the resulting algorithm to specific modal formulae in a straightforward manner, suitable even for implementation on a personal computer. This key idea also improves on previous preliminary work in the same direction (see [S1], [S2]).

§1. Preliminaries. Terminology and notation are essentially as in [TD] (see especially §II.1). Here we recall that a general frame $\mathfrak{F}$ is a triple $(W, r, \mathscr{T})$ in which W is a nonempty set, r is a binary relation on W and $\mathscr{T}$ is a Boolean subalgebra of $\mathscr{P}(W)$ closed under the operation r^* defined by: for every $C \in \mathscr{P}(W)$, $r^*C = \{w \in W\colon \text{for every } v, wrv \text{ implies } v \in C\}$. When $\mathscr{T} = \mathscr{P}(W)$, $\mathfrak{F}$ is called a *Kripke frame*. Given a general frame $\mathfrak{F}$, $\mathfrak{F}^d$ will denote the underlying Kripke frame. As in [TD], we consider $(W, \mathscr{T})$ as a topological space with $\mathscr{T}$ as a base for open (and hence also for closed) subsets; thus we will freely use topological language in relation to general frames whenever useful. For example, a relation r is called *point-closed* if $rw = \{v \in W\colon wrv\}$ is closed for every $w \in W$ (and similarly, r is said to be *closed* if

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$rD = \bigcup\{rw: w \in D\}$ is closed whenever $D \subseteq W$ is closed). So descriptive frames of [G] are just compact Hausdorff frames in which r is point-closed (see [TD, pp. 278, 287]).

Given a valuation V on $\mathfrak{F}$, i.e. a function from propositional variables to $\mathscr{T}$, and a modal formula $A(p_1, \dots, p_k)$, $V(A)$ is the subset of W obtained from $V(p_1), \dots, V(p_k)$ by applying union $\cup$, intersection $\cap$, complementation $-$, and r^* as interpretations in $\mathfrak{F}$ of $\vee$, $\wedge$, $\neg$ and $\Box$, respectively. As a consequence, $\Diamond$ is interpreted as $-r^*-$, which however has a direct mathematical meaning; in fact for every $C \in \mathscr{P}(W)$, $-r^* - C$ is equal to $r^{-1}C = \{w \in W$: there exists v, wrv and $v \in C\}$ where r^{-1} is the inverse of r (see [TD, II.1]). Note that the operations r^* and r^{-1} are often also denoted by the letters l and m respectively.

We write $A(C_1, \dots, C_k)$ as an abbreviation for $V(A(p_1, \dots, p_k))$ where $V(p_i) = C_i$, $i \leq k$, to stress the fact that we conceive the interpretation of a formula A in $\mathfrak{F}$ as an operation with arguments and values in $\mathscr{T}$. Thus, $w \in A(C_1, \dots, C_k)$ is true iff $(\mathfrak{F}, V) \models A[w]$, and hence $(\forall C_1, \dots, C_k \in \mathscr{T})(w \in A(C_1, \dots, C_k))$ is true iff $\mathfrak{F} \models A[w]$. The set-theoretic expression $w \in A(C_1, \dots, C_k)$ is easily transformed into an equivalent first-order formula with second order unary variables $C_1, \dots, C_k$ and just one free individual variable w. In fact, it is enough to replace $w \in r^*A(C_1, \dots, C_k)$ by the equivalent $\forall v(wrv \to v \in A(C_1, \dots, C_k))$, $w \in r^{-1}A(C_1, \dots, C_k)$ by the equivalent $\exists v\{wrv \wedge v \in A(C_1, \dots, C_k)\}$, $w \in (A_1(C_1, \dots, C_k) \cap A_2(C_1, \dots, C_k))$ by the equivalent $w \in A_1(C_1, \dots, C_k) \wedge w \in A_2(C_1, \dots, C_k)$ (and similarly for the other Boolean connectives), and, finally, $w \in C_i$ by $C_i(w)$. Apart from minor notational differences, what we obtain is exactly the well-known standard translation ST(A) of A into a second order formula with the only free individual variable w (see [S], [vB]). We see that, when $\mathfrak{F}$ is a Kripke frame, $\mathfrak{F} \models A[w]$ iff $\mathfrak{F} \models (\forall S_1 \dots S_k)\mathrm{ST}(A)[w]$. The advantage of keeping the expression of the form $w \in A(C_1, \dots, C_k)$, that is of keeping the $\in$ relation "outside", is that such notation allows us to operate not only with classical tautologies, as on any second order formula, but in addition also with mathematical abbreviations, like $\subseteq$, and equivalences such as (a) $w \in r^*C$ iff $rw \subseteq C$, (b) $C_1 \subseteq C$ and $C_2 \subseteq C$ iff $C_1 \cup C_2 \subseteq C$, (c) $(\forall C \in \mathscr{T})(D_1 \subseteq C \to D_2 \subseteq C)$ iff $D_2 \subseteq \bigcap\{C \in \mathscr{T}: D_1 \subseteq C\}, \dots$, which highly simplify the task of manipulating formulae, and, what is most peculiar here, with topological techniques.

Let us see such an approach on an easy example. Let $A(p) = \Box p \to p$; then $A(C)$ will be $r^*C \to C$ (which is an abbreviation for $-r^*C \cup C$), hence $w \in A(C)$ iff $w \in r^*C \to w \in C$ iff $rw \subseteq C \to w \in C$. So, for any general frame $\mathfrak{F}$, $\mathfrak{F} \models \Box p \to p[w]$ iff $(\forall C \in \mathscr{T})(rw \subseteq C \to w \in C)$ iff, by (c) above, $w \in \bigcap\{C \in \mathscr{T}: rw \subseteq C\}$. In §2 we will conclude this example by introducing topological considerations.

We now review the standard definitions used in discussions of modal definability and completeness. A modal formula $A(p_1, \dots, p_k)$ *locally corresponds* to a first order formula $\phi(x)$, with x as the only free variable, if for any Kripke frame $\mathfrak{F}$ and any $w \in W$, $\mathfrak{F} \models A[w]$ iff $\mathfrak{F} \models \phi[w]$; also A *corresponds* to a first order sentence ϕ if for any Kripke frame $\mathfrak{F}$, $\mathfrak{F} \models A$ iff $\mathfrak{F} \models \phi$. Obviously, the former notion implies the latter (see [vB, 3.10]). A modal formula A is (*locally*) *persistent* if for any descriptive frame $\mathfrak{F}$ (and any point $w \in W$), $\mathfrak{F} \models A$ implies $\mathfrak{F}^d \models A$ ($\mathfrak{F} \models A[w]$ implies $\mathfrak{F}^d \models A[w]$); we say that a logic L (here identified with the set of its theses) is persistent when every formula in L is persistent. Now, recall that for every ordinal α the universal general frame $\mathfrak{F}_L(\alpha)$ is a descriptive frame for L (see [TD, I.3, III.2] and [G, §13], where however $\mathfrak{F}_L(\alpha)$ is called a canonical frame). A logic L is *canonical* if for every ordinal α, the Kripke frame $\mathfrak{F}_L(\alpha)^d$, usually called the canonical frame associated to L, is a frame in which L is valid. Then any persistent logic L is clearly canonical (see [vB, 6.10 and 6.11] or [TD, III.4.10], where also the converse is proved). Of course, any canonical logic is complete with respect to Kripke frames.

We finally recall some standard syntactic definitions. A modal formula A is *positive* (*negative*) if it is obtained from (negations of) propositional variables, $\top$ (verum), $\bot$ (falsum), by applying only $\wedge$, $\vee$, $\Diamond$ and $\Box$. A is *monotone in the variable* p_i if $C_i \subseteq D_i$ implies $A(\dots, C_i, \dots) \subseteq A(\dots, D_i, \dots)$. *Positive* and *negative occurrences* of a variable p_i are defined inductively as usual by: (i) p_i is a positive occurrence in p_i, (ii) a positive (negative) occurrence of p_i in A is positive (negative) in $A \vee B$, $A \wedge B$, $\Diamond A$, $\Box A$, and $B \to A$, and negative (positive) in $\neg A$ and $A \to B$. The *dual* $\bar{A}$ of A is obtained from A by interchanging $\top$ with $\bot$, $\vee$ with $\wedge$, $\Box$ with $\Diamond$. We collect some facts about the above definitions in the following lemma, which is easily proved by induction.

LEMMA 1.1. *Let $A(p_1, \dots, p_k)$ be any modal formula written without $\to$. Then:*

(i) *A is equivalent to a positive (negative) formula if and only if all the variables occur only positively (negatively) in A.*

(ii) *$\neg A(p_1, \dots, p_k)$ is equivalent to $\bar{A}(\neg p_1, \dots, \neg p_k)$.*

(iii) *A is equivalent to a formula in which each occurrence of $\neg$, if any, is only in front of variables.*

(iv) *If p_i occurs only positively in A, then A is monotone in p_i.*

§2. A topological proof of Sahlqvist's theorem. We first take up again the formula $\Box p \to p$ to illustrate on an example the key topological idea of the proof. Observe that $\bigcap\{C \in \mathscr{T}: rw \subseteq C\}$ is by definition simply the closure $\overline{rw}$ of rw in $\mathscr{T}$. So assume $\mathfrak{F}$ to be a descriptive frame; then r is point closed, i.e. $\overline{rw} = rw$ for every $w \in W$, and hence $\mathfrak{F} \models \Box p \to p[w]$ iff $w \in rw$, that is $\mathfrak{F}^d \models Rxx[w]$. Since the same argument continues to hold also when $\mathfrak{F}$ is a Kripke frame (because rw is then trivially closed), $\Box p \to p$ locally corresponds to reflexivity (and this is indeed the fastest proof of this fact that we know of). But the proper use of the above argument on descriptive frames is to obtain persistence (and hence completeness) quite easily: for any descriptive frame $\mathfrak{F}$, from $\mathfrak{F} \models \Box p \to p[w]$ we reach $\mathfrak{F}^d \models Rxx[w]$, and then go back to $\mathfrak{F}^d \models \Box p \to p[w]$ by correspondence. We believe that a reader who tries the same method on, for example, $p \to \Box p$, $\Box p \to \Box^2 p$, $p \wedge \Box p \wedge \Box^2 p \wedge \cdots \wedge \Box^n p \to \Box^{n+1} p$, will appreciate its convenience over previous more lengthy proofs.

Now, our purpose is to show that the class of Sahlqvist formulae can be seen just as the biggest class to which the same tricks, with some combinatorial complications, can be applied. Note that in the above example the solution was reached when we found a single subset rw, for which $w \in rw$ is first order expressible, and which, keeping logical equivalence, can be substituted for the second order variable C and its quantification. The role played by rw in the above example is taken in the general case by the expressions of the form $r^{m_1}w_1 \cup \cdots \cup r^{m_h}w_h$, $h \geq 0$, $m_h \geq 0$, which we here call *r-expressions* (note that, by convention, $r^0 w = \{w\}$ and $r^{m+1}w = r(r^m w)$). When a general frame $\mathfrak{F}$ and $w, w_1, \dots, w_h \in W$ are fixed, any r-expression

D will be identified with the subset of W it defines; it is clear, however, that $w \in D$ is equivalently expressed by a first order formula, which does not depend on $\mathfrak{F}$ (for example, $w \in \{w_1\}$ is equivalent to $w = w_1$ and $w \in r^m w_1 \cup r^n w_2$ is equivalent to

$$\exists v_1 \cdots \exists v_{m-1}(w_1 r v_1 \wedge \cdots \wedge v_{m-1} r w) \vee \exists v'_1 \cdots \exists v'_{n-1}(w_2 r v'_1 \wedge \cdots \wedge v'_{n-1} r w);$$

note that when $h = 0$, D is the empty set and hence $w \in D$ is equivalent to $w \neq w$). Combining this with the remarks of §1, we obtain as an immediate consequence that:

LEMMA 2.1. *Let $A(p_1, \ldots, p_k)$ be any modal formula. Then, for every r-expression $D_1, \ldots, D_k$, $w \in A(D_1, \ldots, D_k)$ is equivalent to a first order formula which has w among its free variables.*

Note that a much wider class of expressions would satisfy Lemma 2.1 (for example, we could include $r^{-1}, \cap, -$); the reason for restricting the definition as we did above is that when $\mathfrak{F}$ is a descriptive frame (or a Kripke frame), every r-expression is a closed subset of W, since r is closed (see [TD, II.2.12]).

Now, the r-expressions are strictly related to the following formulae:

DEFINITION 2.2. A modal formula A is *strongly positive* if it is obtained from formulae of the form $\Box^m p$ $(m \geq 0)$ by applying only $\wedge$.

STRONGLY POSITIVE LEMMA. *Let $A(p_1, \ldots, p_k)$ be a strongly positive formula and $\mathfrak{F}$ any general frame. Then for every $w \in W$ there exist r-expressions $D_1, \ldots, D_k$ such that for every $C_1, \ldots, C_k \in \mathcal{T}$,*

$$w \in A(C_1, \ldots, C_k) \quad \textit{iff} \quad D_1 \subseteq C_1 \wedge \cdots \wedge D_k \subseteq C_k.$$

PROOF. Consider any conjunct of the form $\Box^m p$; then $w \in (r^*)^m C$ is equivalent to $w \in (r^m)^* C$, by II.2.1 of [TD], and hence, by definition, also to $r^m w \subseteq C$. Now, by using the mathematical equivalence (b) of §1, we can reduce to one the number of occurrences of each C_i after an inclusion symbol.

It is now an easy exercise (which we leave to the reader, but it is not used below) to prove that the wide class of modal formulae of the form $A_1 \to A_2$, with $A_i, i \leq 2$, strongly positive, can be dealt with by the trick of the example. This class immediately grows after the intersection lemma below. To prove it, we need a result first stated in [E]:

ESAKIA'S LEMMA. *Let $\mathfrak{F}$ be any descriptive frame. Then for each downward directed family $\mathscr{C} = \{C_i\}_{i \in I}$ of nonempty closed subsets of W, $r^{-1}(\bigcap_{i \in I} C_i) = \bigcap_{i \in I} r^{-1} C_i$.*

PROOF. Since r^{-1} is monotone, $r^{-1}(\bigcap_{i \in I} C_i) \subseteq \bigcap_{i \in I} r^{-1} C_i$. In order to prove the converse, let $w \in \bigcap_{i \in I} r^{-1} C_i$. Then $w \in r^{-1} C_i$ for every $i \in I$, that is, $rw \cap C_i \neq \varnothing$ for every $i \in I$. Then $\{rw\} \cup \mathscr{C}$ has the finite intersection property, and so, by compactness, $rw \cap (\bigcap_{i \in I} C_i) \neq \varnothing$, from which $w \in r^{-1}(\bigcap_{i \in I} C_i)$.

By Esakia's lemma and the definition of closure, r^{-1} is closed on any descriptive frame $\mathfrak{F}$. Since r^* distributes over intersections on any frame (see [TD, II.1.8 and II.1.10]), r^* is also closed. So for any positive formula A and any descriptive frame $\mathfrak{F}$, $A(D_1, \ldots, D_k)$ is a closed subset of W whenever $D_1, \ldots, D_k \subseteq W$ are closed (note that here we consider $A(-, \ldots, -)$ just as a set theoretical operation, independently of interpretations).

INTERSECTION LEMMA. *Let $\mathfrak{F}$ be a Kripke or a descriptive frame and let A be a positive formula. Then, for each $D \subseteq W$,*

$$A(\ldots,\bar{D},\ldots) = \bigcap\{A(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\}$$

where $\bar{D} = \bigcap\{C \in \mathcal{T}: D \subseteq C\}$ is the closure of D in $\mathcal{T}$ and the other parameters range over closed subsets of W.

PROOF. When $\mathfrak{F}$ is a Kripke frame the claim is trivial since A is monotone and $\bar{D} = D$.

When $\mathfrak{F}$ is a descriptive frame, the proof is by induction on the complexity of A.

The case $A = p$ is exactly the definition of closure.

Let $A = A_1 \vee A_2$. To prove the inclusion

$$\bigcap\{(A_1 \vee A_2)(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\} \subseteq (A_1 \vee A_2)(\ldots,\bar{D},\ldots),$$

let $w \notin A_1(\ldots,\bar{D},\ldots) \cup A_2(\ldots,\bar{D},\ldots)$; then there exist $C', C'' \in \mathcal{T}$ with $D \subseteq C'$ and $D \subseteq C''$ such that $w \notin A_1(\ldots,C',\ldots)$ and $w \notin A_2(\ldots,C'',\ldots)$. Now, since A_1 and A_2 are positive formulae, by Lemma 1.1.iv, $w \notin A_1(\ldots,C' \cap C'',\ldots)$ and $w \notin A_2(\ldots,C' \cap C'',\ldots)$, from which

$$w \notin \bigcap\{(A_1 \vee A_2)(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\}.$$

Thus we have proved

$$\bigcap\{(A_1 \vee A_2)(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\} \subseteq (A_1 \vee A_2)(\ldots,\bar{D},\ldots).$$

The other inclusion follows from routine set-theoretical facts. When $A = A_1 \wedge A_2$, the proof is quite similar.

The case $A = \Box A_1$ follows immediately since, as recalled above, r^* distributes over intersections.

Let $A = \Diamond A_1$. Since A_1 is positive, by Lemma 1.1.iv and the remarks following Esakia's lemma, $\mathcal{C} = \{A(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\}$ is a downward directed family of closed subsets. The claim now follows from the inductive hypothesis by Esakia's lemma.

As a generalization of (c) of §1, we see that when A_2 is positive,

$$w \in \bigcap\{A_2(\ldots,C,\ldots): D \subseteq C \in \mathcal{T}\} \quad \text{iff} \quad (\forall C \in \mathcal{T})(D \subseteq C \to w \in A_2(\ldots,C,\ldots)),$$

which explains the crucial role of the intersection lemma: the elimination of a quantified second order variable C in favour of a single subset. So, if we consider $A = (A_1 \to A_2)(p_1,\ldots,p_k)$ with A_1 strongly positive and A_2 positive, we see that

$$(\forall C_1 \cdots C_k \in \mathcal{T})(w \in A_1(C_1,\ldots,C_k) \to w \in A_2(C_1,\ldots,C_k))$$

iff

$$(\forall C_1 \cdots C_k \in \mathcal{T})(D_1 \subseteq C_1 \wedge \cdots \wedge D_k \subseteq C_k \to w \in A_2(C_1,\ldots,C_k))$$

(by the strongly positive lemma) iff

$$(\forall C_1 \cdots C_{k-1} \in \mathcal{T})\left(\bigwedge_{i \leq k-1} D_i \subseteq C_i \to (\forall C_k \in \mathcal{T})(D_k \subseteq C_k \to w \in A_2(C_1,\ldots,C_k))\right)$$

iff

$$(\forall C_1 \cdots C_{k-1} \in \mathscr{T})\left(\bigwedge_{i \leq k-1} D_i \subseteq C_i \rightarrow w \in A_2(C_1, \ldots, C_{k-1}, D_k)\right),$$

by the above-mentioned generalization of (c), the intersection lemma and the fact that D_k is closed. Repeating this $k - 1$ times, we obtain:

LEMMA 2.3. *Let* $\mathfrak{F}$ *be a Kripke or a descriptive frame and*

$$A = (A_1 \rightarrow A_2)(p_1, \ldots, p_k)$$

where A_1 *is strongly positive and* A_2 *is positive. Then,*

$$(\forall C_1 \cdots C_k \in \mathscr{T})(w \in A(C_1, \ldots, C_k)) \quad \textit{iff} \quad w \in A_2(D_1, \ldots, D_k),$$

where D_i, $i \leq k$, *are suitable r-expressions, effectively obtainable from* A.

The further step is to find antecedents which, in a sense, can be decomposed in such a way that we can apply the intersection lemma in a similar way. This is achieved in Lemma 2.5 below.

DEFINITION 2.4. A modal formula A is an *untied* formula if it is obtained from strongly positive formulae and negative ones by applying only $\wedge$ and $\Diamond$.

LEMMA 2.5. *Let* $A(p_1, \ldots, p_k)$ *be any untied formula. Then* $w \in A(C_1, \ldots, C_k)$ *is equivalent to*

$$\exists v_1 \cdots \exists v_n\left(\psi \wedge \bigwedge_{i \leq k} D_i \subseteq C_i \wedge \bigwedge_{j \leq m} u_j \in N_j(C_1, \ldots, C_k)\right)$$

where the v_i, $i \leq n$, *are distinct variables different from* w, ψ *is a conjunction of atomic formulae of the form* $u_i r u_j$ *and all the* u*'s are among* $v_1, \ldots, v_n$ *and* w, *the* D_i, $i \leq k$, *are suitable r-expressions, and the* N_j, $j \leq m$, *are negative formulae and all indices may be zero.*

PROOF. The proof is by induction on the complexity of A. It is trivial in the case in which A is strongly positive or negative. In the other cases, the claim follows by changing names to variables if necessary, by using the equivalence (b) and by shifting quantifiers.

MAIN LEMMA. *Let* $A = \Box^m(A_1 \rightarrow A_2)(p_1, \ldots, p_k)$, *where* A_1 *is an untied formula,* A_2 *is positive and* $m \geq 0$. *Then* A *is locally persistent and locally corresponds to a first order formula* $\phi(x)$ *effectively obtainable from* A.

PROOF. Let w be any point of a descriptive frame $\mathfrak{F}$. Then

(1) $$(\forall C_1 \cdots C_k \in \mathscr{T})(w \in \Box^m(A_1 \rightarrow A_2)(C_1, \ldots, C_k))$$

iff

(2) $$(\forall C_1 \cdots C_k \in \mathscr{T})(\forall v)(w r^m v \rightarrow v \in (A_1 \rightarrow A_2)(C_1, \ldots, C_k))$$

iff, by Lemma 2.5,

(3) $$(\forall C_1 \cdots C_k \in \mathscr{T})(\forall v)\Bigg(w r^m v \rightarrow \Bigg((\exists v_1 \cdots \exists v_n)$$
$$\left(\psi \wedge \bigwedge_{i \leq k} D_i \subseteq C_i \wedge \bigwedge_{j \leq m} u_j \in N_j(C_1, \ldots, C_k)\right) \rightarrow v \in A_2(C_1, \ldots, C_k)\Bigg)\Bigg).$$

Then, by a shift of quantifiers and classical logic, (3) can be rewritten as

(4) $(\forall C_1 \cdots C_k \in \mathscr{T})(\forall v)(\forall v_1 \cdots \forall v_n)$

$$\left(\psi' \rightarrow \left(\left(\bigwedge_{i \leq k} D_i \subseteq C_i \wedge \bigwedge_{j \leq m} u_j \in N_j(C_1, \ldots, C_k)\right) \rightarrow v \in A_2(C_1, \ldots, C_k)\right)\right)$$

where $\psi' = wr^m v \wedge \psi$. By Lemma 1.1.ii, $\neg(\bigwedge_{j \leq m} u_j \in N_j(C_1, \ldots, C_k))$ is equivalent to $\bigvee_{j \leq m} u_j \in \bar{N}_j(\neg C_1, \ldots, \neg C_k)$ and $\bar{N}_j(\neg p_1, \ldots, \neg p_k)$ is positive. So putting $P_j = \bar{N}_j(\neg p_1, \ldots, \neg p_k)$, $P_{m+1} = A_2(p_1, \ldots, p_k)$ and $u_{m+1} = v$, (4) is equivalent to

$$(5) \qquad (\forall v)(\forall v_1 \cdots \forall v_n)\left(\psi' \rightarrow \left((\forall C_1 \cdots C_k \in \mathscr{T}) \left(\bigwedge_{i \leq k} D_i \subseteq C_i \rightarrow \bigvee_{j \leq m} u_j \in P_j(C_1, \ldots, C_k)\right)\right)\right).$$

By k applications of the intersection lemma, since D_i, $i \leq k$, is a closed subset of W, it follows that

$$(6) \qquad (\forall v)(\forall v_1 \cdots \forall v_n)\left(\psi \rightarrow \bigvee_{j \leq m} u_j \in P_j(D_1, \ldots, D_k)\right).$$

Since $D_1, \ldots, D_k$ are r-expressions, $P_j(D_1, \ldots, D_k)$, $j \leq m + 1$, is first order expressible by Lemma 2.1; so now it is only a matter of transcribing the first order formula $\phi(x)$ which makes (6) equivalent to $\mathfrak{F}^d \models \phi[w]$. Note that every step of the above argument goes through also when $\mathfrak{F}$ is a Kripke frame; we thus have proved that A locally corresponds to $\phi(x)$. But now, starting from $\mathfrak{F} \models A[w]$ when $\mathfrak{F}$ is a descriptive frame, we arrive at $\mathfrak{F}^d \models \phi[w]$ as above, and we go back to $\mathfrak{F}^d \models A[w]$ by local correspondence: the result is local persistence.

In order to prove Sahlqvist's theorem, we now only need a last syntactical lemma that is easily provable by induction.

LEMMA 2.6. *Let $A(p_1, \ldots, p_k)$ be any modal formula written without $\rightarrow$. Then A is equivalent to a disjunction of untied formulae iff, after rewriting A as in Lemma* 1.1.iii, *no positive occurrence of p_i in A, $i \leq k$, is in a subformula of the form $B_1 \vee B_2$ or $\Diamond B_1$ within the scope of some $\Box$.*

PROOF OF SAHLQVIST'S THEOREM. By Lemma 2.6 above, any A_1 satisfying the assumptions is equivalent to a disjunction $B_1 \vee \cdots \vee B_h$, where each B_i is untied. Hence, $\Box^m(A_1 \rightarrow A_2)$ is equivalent to $\Box^m(B_1 \rightarrow A_2) \wedge \cdots \wedge \Box^m(B_h \rightarrow A_2)$. By the main lemma, each conjunct $\Box^m(B_i \rightarrow A_2)$ is persistent and corresponds to a first order formula; then also $\Box^m(A_1 \rightarrow A_2)$ and A have such properties, since they are obviously preserved by conjunction and equivalence. Finally, as we recalled in §1, $L + A$ is canonical whenever L is a canonical logic and A is persistent.

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