A few results about topological types - Definitions and bibliography

Definition 1 (ordinal) *A set S is an ordinal if and only if it satisfies the two following conditions :*

(O_1) *every non-empty subset of S has a least element for the relation $\in$;*[1]

(O_2) *for all x, if $x \in S$ then $x \subseteq S$.*

Definition 2 (inverse image) *Let $\mathbb{U}$ and $\mathbb{V}$ be sets ; the inverse image of $S \subseteq \mathbb{V}$ under the function $f : \mathbb{U} \to \mathbb{V}$ is the subset $f^{-1}[S]$ of $\mathbb{U}$ defined as follows :*

$$f^{-1}[S] := \{u \in \mathbb{U} \ ; \ f(u) \in S\}.$$

Definition 3 (topology) *Let $\mathbb{U}$ be a set ; $\mathcal{T} \subseteq \wp(\mathbb{U})$ is a topology on $\mathbb{U}$ if and only if the three following conditions are satisfied :*

(T_1) *$\varnothing \in \mathcal{T}$ and $\mathbb{U} \in \mathcal{T}$;*

(T_2) *$\mathcal{T}$ is closed under countable union ;*

(T_3) *$\mathcal{T}$ is closed under finite intersection.*

Definition 4 (base) *Let $\mathbb{U}$ be a set and $\mathcal{T}$ a topology on $\mathbb{U}$; $\mathcal{B} \subseteq \wp(\mathbb{U})$ is a base for $\mathcal{T}$ if and only if every non-empty element of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.*

Definition 5 (open set, closed set, clopen set) *Let $\mathbb{U}$ be a set and $\mathcal{T}$ a topology on $\mathbb{U}$; an open set is an element of $\mathcal{T}$, a closed set is an element of $\{\complement_{\mathbb{U}}^{O} \in \wp(\mathbb{U}) \,;\, O \in \mathcal{T}\}$ where $\complement_{\mathbb{U}}^{O} := \mathbb{U} \ominus O := \{x \in \mathbb{U} \,;\, x \notin O\}$, and a clopen set is an element of $\mathcal{T} \cap \{\complement_{\mathbb{U}}^{O} \in \wp(\mathbb{U}) \,;\, O \in \mathcal{T}\}$.*

Definition 6 (topological space) T *is a topological space if and only if* $\mathrm{T} := (\mathbb{U}, \mathcal{T})$ *where $\mathbb{U}$ is a set and $\mathcal{T}$ is a topology on $\mathbb{U}$.*

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[1] In other words, S is well-ordered by the relation $\in$.

Definition 7 (continuous function) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$ *be topological spaces ;* $f : \mathbb{U} \to \mathbb{V}$ *is a continuous function if and only if for all* $O \in \mathcal{T}_{\mathbb{V}}$*,* $f^{-1}[O] \in \mathcal{T}_{\mathbb{U}}$*.*

Definition 8 (homeomorphism) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$ *be topological spaces ; a function* $f : \mathbb{U} \to \mathbb{V}$ *is a homeomorphism between* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$ *if and only if the three following conditions are satisfied :*

(H_1) f *is a bijection ;*

(H_2) f *is continuous ;*

(H_3) f^{-1} *is continuous.*

Definition 9 (homeomorphic) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$ *be topological spaces ;* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$ *are homeomorphic, noted* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}}) \cong (\mathbb{V}, \mathcal{T}_{\mathbb{V}})$*, if and only if there exists an homeomorphism* f *between* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *and* $(\mathbb{V}, \mathcal{T}_{\mathbb{V}})$*.*

Definition 10 (topological type) *Let* $\mathscr{T}$ *be the class of all topological spaces ; a topological type is an element of* $\mathscr{T} / \cong$*.*

Definition 11 (closure) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *be a topological space and* $S \in \wp(\mathbb{U})$ *; the closure of* S*, noted* $\mathrm{CL}(S)$*, is the smallest closed set containing* S*.*

Definition 12 (accumulation point) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *be a topological space and* $S \in \wp(\mathbb{U})$ *;* $x \in \mathbb{U}$ *is an accumulation point of* S *if and only if* $x \in \mathrm{CL}(S \ominus \{x\})$*.*

Definition 13 (derivative) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *be a topological space and* $S \in \wp(\mathbb{U})$ *; the derivative of* S*, noted* S'*, is the set of all accumulation points of* S*.*

Definition 14 (transfinite derivative of degree α) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *be a topological space and* $S \in \wp(\mathbb{U})$ *; the transfinite derivative of degree* α *of* S *where* α *is an ordinal, noted* $S^{(\alpha)}$*, is defined as follows :*

$$S^{(\alpha)} = \begin{cases} S & if\ \alpha = 0 \\ \left(S^{(\beta)}\right)' & if\ \alpha = \beta + 1 \\ \bigcap\limits_{\delta < \lambda} S^{(\delta)} & if\ \alpha = \lambda \text{ with } \lambda \text{ limit.} \end{cases}$$

Definition 15 ((topological) neighbourhood) *Let* $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ *be a topological space and* $x \in \mathbb{U}$ *;* $N \in \wp(\mathbb{U})$ *is a neighbourhood of* x *if and only if there exists* $O \in \mathcal{T}_{\mathbb{U}}$ *such that* $x \in O$ *and* $O \subseteq N$*.*

Definition 16 ((topological) neighbourhood system) *Let $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ be a topological space and $x \in \mathbb{U}$; the neighbourhood system of x, noted $\nu(x)$, is the set of all its neighbourhoods.*[2]

Definition 17 ((topological) neighbourhood base) *Let $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ be a topological space and $x \in \mathbb{U}$; a neighbourhood base for x, noted $\mathrm{b}(x)$, is a subset of $\nu(x)$ such that for all elements $V \in \nu(x)$ there exists $B \in \mathrm{b}(x)$ such that $B \subseteq V$.*

Definition 18 (dispersed set) *A set S is dispersed if and only if it does not contain any set $X \neq \varnothing$ such that $X \subseteq X'$.*

Definition 19 (limit complexity, coefficient, purity) *Let α be an ordinal whose (unique) Cantor normal form is*

$$\omega^{\alpha_0} \cdot k_0 + \ldots + \omega^{\alpha_n} \cdot k_n$$

where $\alpha \geq \alpha_0 > \ldots > \alpha_n$ and $0 < k_i < \omega$ for $0 \leq i \leq n$; the limit complexity of α, noted $\mathrm{lc}(\alpha)$, is α_0; the coefficient of α, noted $\mathrm{c}(\alpha)$, is k_0; the purity of α, noted $\mathrm{p}(\alpha)$, is defined as follows :

$$\mathrm{p}(\alpha) := \begin{cases} 0 & \textit{if } \alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) \textit{ and } \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) \geq \omega \\ \omega^{\alpha_n} \cdot k_n & \textit{if } \alpha \neq \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) \textit{ or } \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) < \omega. \end{cases}$$

Definition 20 (Cantor-Bendixon rank) *Let S be a set, the Cantor-Bendixon rank of $x \in S$ is $\mathrm{CB}_S(x) := \sup\{\alpha \in \mathrm{On}\,;\, x \in S^{(\alpha)}\}$.*

Definition 21 (limit point) *Let $(\mathbb{U}, \mathcal{T}_{\mathbb{U}})$ be a topological space and $S \subseteq \mathbb{U}$; $x \in \mathbb{U}$ is a limit point of S if and only if every element of $\mathcal{T}_{\mathbb{U}}$ that contains x also contains a point of S that has to be different from x.*

Definition 22 (cofinal subset) *Let S be a set partially ordered*[3] *by $\leq_S$ and $C \subseteq S$; C is a cofinal subset of S if and only if for all $x \in S$ there exists a $y \in C$ such that $x \leq_S y$.*

Definition 23 ($\langle C, \alpha \rangle$-slope) *Let η be a limit ordinal*[4]*, α an arbitrary ordinal and $C \subseteq \eta$ a confinal subset of η; a function $f : C \to \alpha$ is a $\langle C, \alpha \rangle$-slope if and only if $\mathrm{CB}_\alpha(f(\gamma)) = \gamma$ for all $\gamma \in C$.*

[2] A topological neighbourhood system is also called a "topological neighbourhood filter".

[3] A partial order is a binary relation R over a set P which is reflexive, antisymmetric, and transitive.

[4] A limit ordinal is an ordinal that is neither zero nor successor.

Definition 24 (top) *Let η be a limit ordinal, α an arbitrary ordinal, $C \subseteq \eta$ a confinal subset of η and $f : C \to \alpha$ a $\langle C, \alpha\rangle$-slope; $\tau \in \alpha$ is a top of the slope f if and only if for every $\delta < \eta$, the ordinal τ is a limit point of the set $f_\delta := \{f(\gamma)\,;\, \gamma \in C_\delta\}$ where $C_\delta := \{x \in C\,;\, x > \delta\}$.*

Definition 25 (cofinal slope, supremum) *Let η be a limit ordinal, α an arbitrary ordinal, $C \subseteq \eta$ a confinal subset of η, $f : C \to \alpha$ a $\langle C, \alpha\rangle$-slope and $\delta < \eta$; a function $g : \delta \to \sigma_\delta$ where $\sigma_\delta := \bigcup\{f(\gamma)\,;\, \gamma \in C_\delta\}$ for $C_\delta := \{x \in C\,;\, x > \delta\}$ is a cofinal slope if and only if g is a constant function whose constant value is the supremum of f.*

Definition 26 (σ-algebra) *Let $\mathbb{U}$ be a set; $\mathcal{S} \subseteq \mathscr{P}(\mathbb{U})$ is a σ-algebra on $\mathbb{U}$ if and only if the three following conditions are satisfied :*

(A_1) *$\mathcal{S} \neq \emptyset$;*

(A_2) *$\mathcal{S}$ is closed under complementation;*

(A_3) *$\mathcal{S}$ is closed under countable union.*

Definition 27 (generated σ-algebra) *Let $\mathbb{U}$ be a set and $S \subseteq \mathscr{P}(\mathbb{U})$; $\sigma(S)$ is the generated σ-algebra on $\mathbb{U}$ from S if and only if it is the intersection of all the σ-algebras on $\mathbb{U}$ that contain S.*

Definition 28 (Borel σ-algebra) *Let $(\mathbb{U}, \mathcal{T}_\mathbb{U})$ be a topological space; the Borel σ-algebra on $(\mathbb{U}, \mathcal{T}_\mathbb{U})$, noted $\mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U})$, is the generated σ-algebra on $\mathbb{U}$ from $\mathcal{T}_\mathbb{U}$, that is $\mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U}) = \sigma(\mathcal{T}_\mathbb{U})$.*

Definition 29 (Borel isomorphism) *Let $(\mathbb{U}, \mathcal{T}_\mathbb{U})$ and $(\mathbb{V}, \mathcal{T}_\mathbb{V})$ be topological spaces whose respective Borel σ-algebras are $\mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U})$ and $\mathcal{B}_\mathbb{V}(\mathcal{T}_\mathbb{V})$; $f : \mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U}) \to \mathcal{B}_\mathbb{V}(\mathcal{T}_\mathbb{V})$ is a Borel isomorphism between $(\mathbb{U}, \mathcal{T}_\mathbb{U})$ and $(\mathbb{V}, \mathcal{T}_\mathbb{V})$ if and only if the three following conditions are satisfied :*

(B_1) *f is a bijection;*

(B_2) *for every $U \in \mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U})$, $f[U] \in \mathcal{B}_\mathbb{V}(\mathcal{T}_\mathbb{V})$;*

(B_3) *for every $V \in \mathcal{B}_\mathbb{V}(\mathcal{T}_\mathbb{V})$, $f^{-1}[V] \in \mathcal{B}_\mathbb{U}(\mathcal{T}_\mathbb{U})$.*

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