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Annals of Pure and Applied Logic 131 (2005) 287–301
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Completeness of S4 with respect to the real line: revisited[☆]

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Received 1 October 2001; received in revised form 30 November 2003; accepted 3 June 2004

Available online 2 September 2004

Communicated by S.N. Artemov

Abstract

We prove that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of the real line, thus strengthening a 1944 result of McKinsey and Tarski (Ann. of Math. (2) 45 (1944) 141). We also prove that the same result holds for the bimodal system **S4** + **S5** + **C**, which is a strengthening of a 1999 result of Shehtman (J. Appl. Non-Classical Logics 9 (1999) 369).

MSC (1991): 03B45; 06E25; 54C10

Keywords: Modal logic; Topology; Boolean algebras with operators; Completeness

1. Introduction

It was shown in McKinsey and Tarski [8] that every finite well-connected topological space is an open image of a metric separable dense-in-itself space. This together with the finite model property of **S4** implies that **S4** is complete with respect to any metric separable dense-in-itself space. Most importantly, it implies that **S4** is complete with respect to the real line $\mathbb{R}$. Shehtman [13] strengthened the McKinsey and Tarski result by showing that

☆ The research described in this publication was made possible in part by Award No. 3303 of the Georgian Research and Development Foundation (GRDF) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).
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doi:10.1016/j.apal.2004.06.003

every finite connected space is an open image of a (connected) metric separable dense-in-itself space. (That every finite connected space is an open image of a Euclidean space was first established in Puckett [11].) As a result, Shehtman obtained that in the language enriched with the universal modality $\forall$ the complete logic of a connected metric separable dense-in-itself space is the logic **S4** + **S5** + **C**, where **S4** + **S5** is Bennett's logic [2] (being **S4** for $\Box$, **S5** for $\forall$, plus the bridge axiom $\forall\varphi \to \Box\varphi$) and **C** is the connectedness axiom $\forall(\Diamond\varphi \to \Box\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$.

The original proof of McKinsey and Tarski was quite complicated. The later version in Rasiowa and Sikorski [12] was not much more accessible. Recently Mints [10] and Aiello et al. [1] obtained simpler model-theoretic proofs of completeness of **S4** with respect to the Cantor space $\mathcal{C}$ and the real line $\mathbb{R}$. In this paper we give yet another, more topological, proof of completeness of **S4** with respect to $\mathbb{R}$. It is not only more accessible than the original proof, but also strengthens both the McKinsey and Tarski, and Shehtman results.

The paper is organized as follows. In Section 2 we recall a one-to-one correspondence between Alexandroff spaces and quasi-ordered sets; we also recall the modal systems **S4**, **S4** + **S5** and **S4** + **S5** + **C**, and their algebraic semantics. In Section 3 we give a simplified proof that a finite well-connected topological space is an open image of $\mathbb{R}$. It follows that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbb{R}$, which is a strengthening of the McKinsey and Tarski result. As a by-product, we obtain a new proof of completeness of the intuitionistic propositional logic **Int** with respect to open subsets of $\mathbb{R}$, and completeness of the Grzegorczyk logic **Grz** with respect to Boolean combinations of open subsets of $\mathbb{R}$. In Section 4 we give a simplified proof that a finite topological space is an open image of $\mathbb{R}$ iff it is connected. Consequently, we obtain that **S4** + **S5** + **C** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbb{R}$, which is a strengthening of the Shehtman result. We conclude the paper by mentioning several open problems.

2. Preliminaries

2.1. Topology and order

Suppose X is a topological space. For $A \subseteq X$ we denote by $\overline{A}$ the closure of A, and by $Int(A)$ the interior of A. We recall that A is *dense* if $\overline{A} = X$, and that A is *nowhere dense* or *boundary* if $Int(A) = \emptyset$. The definition of closed and open subsets of X is usual. We call a subset of X *clopen* if it is simultaneously closed and open. The space X is called *connected* if $\emptyset$ and X are the only clopen subsets of X; it is called *well-connected* if there exists a least nonempty closed subset of X. It is obvious that every well-connected space is connected, but the converse is not necessarily true. We call X an *Alexandroff space* if the intersection of any family of open subsets of X is open. Obviously every finite space is an Alexandroff space. For two topological spaces X and Y, a continuous map $f : X \to Y$ is called *open* if the f-image of every open subset of X is an open subset of Y. Thus, f is an open map iff it *preserves* and *reflects* opens.

Suppose X is a nonempty set. A binary relation $\leq$ on X is called a *quasi-order* if $\leq$ is reflexive and transitive; if in addition $\leq$ is antisymmetric, then $\leq$ is called a *partial order*. If $\leq$ is a quasi-order on X, then X is called a *quasi-ordered set* or simply a *qoset*; if $\leq$ is a partial order, then X is called a *partially ordered set* or simply a *poset*. For two qosets X and Y, an order-preserving map $f : X \to Y$ is called a *p-morphism* if for every $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there exists $z \in X$ such that $x \leq z$ and $f(z) = y$.

Suppose X is a qoset. For $A \subseteq X$ let $\uparrow A = \{x \in X : \exists a \in A \text{ with } a \leq x\}$ and $\downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\}$. We call $A \subseteq X$ an *upset* if $A = \uparrow A$, and a *downset* if $A = \downarrow A$. For $x \in X$ let $C[x] = \{y \in X : x \leq y \text{ and } y \leq x\}$. We call $C \subseteq X$ a *cluster* if there is $x \in X$ such that $C = C[x]$. We call $x \in X$ *maximal* if $x \leq y$ implies $x = y$, and *quasi-maximal* if $x \leq y$ implies $y \leq x$; similarly, we call $x \in X$ *minimal* if $y \leq x$ implies $y = x$, and *quasi-minimal* if $y \leq x$ implies $x \leq y$. If X is a poset, then it is obvious that the notions of maximal and quasi-maximal points, as well as the notions of minimal and quasi-minimal points coincide. We call a cluster C *maximal* if $C = C[x]$ for some quasi-maximal $x \in X$; a cluster C is called *minimal* if $C = C[x]$ for some quasi-minimal $x \in X$. We call $r \in X$ a *root* of X if $r \leq x$ for every $x \in X$; a qoset X is called *rooted* if it has a root r; note that r is not unique: every element of $C[r]$ serves as a root of X. We say that there exists a *$\leq$-path* between two points x, y of X if there exists a sequence $w_1, \ldots, w_n$ of points of X such that $w_1 = x$, $w_n = y$, and either $w_i \leq w_{i+1}$ or $w_{i+1} \leq w_i$ for any $1 \leq i \leq n - 1$. We call X a *connected component* if there is a $\leq$-path between any two points of X. Note that every rooted qoset is a connected component, but not vice versa.

For a qoset X let $\tau_{\leq}$ denote the set of upsets of X. It is easy to verify that $\tau_{\leq}$ is an Alexandroff topology on X. Conversely, if X is a topological space, then we define the *specialization order* $\leq_\tau$ on X by putting $x \leq_\tau y$ iff $x \in \overline{\{y\}}$. It is routine to check that $\leq_\tau$ is a quasi-order on X. Moreover, $\leq_\tau$ is a partial order iff X is a T_0-space. Now a standard argument shows that $\leq = \leq_{\tau_{\leq}}$ and that $\tau \subseteq \tau_{\leq_\tau}$. Furthermore, $\tau = \tau_{\leq_\tau}$ iff τ is an Alexandroff topology. This establishes a one-to-one correspondence between qosets and Alexandroff spaces, and between posets and Alexandroff T_0-spaces. In particular, we obtain a one-to-one correspondence between finite qosets and finite topological spaces, and between finite posets and finite T_0-spaces. We note that under this correspondence order-preserving maps correspond to continuous maps, and p-morphisms correspond to open maps. Moreover, connected spaces correspond to connected components and well-connected spaces correspond to rooted qosets (see, e.g., Aiello et al. [1] for details).

Subsequently, we will not distinguish between Alexandroff spaces and qosets, and between Alexandroff T_0-spaces and posets. For these spaces we will use interchangeably the notions of open maps and p-morphisms, connected spaces and connected components, and well-connected spaces and rooted qosets.

2.2. **S4**, **S4** + **S5**, *and* **S4** + **S5** + **C**

We recall that **S4** is the least set of formulae of the propositional modal language $\mathcal{L}$ containing the axioms $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$, $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and closed under modus ponens $(\varphi, \varphi \to \psi/\psi)$, substitution $(\varphi(p_1, \ldots, p_n)/\varphi(\psi_1/p_1, \ldots, \psi_n/p_n))$, and necessitation $(\varphi/\Box\varphi)$.

It was shown in McKinsey and Tarski [9] that algebraic models of **S4** are closure algebras. We recall that a *closure algebra* is a pair (B, C), where B is a Boolean algebra and

$C : B \to B$ is a function satisfying the following identities: (i) $a \leq Ca$, (ii) $CCa = Ca$, (iii) $C(a \vee b) = Ca \vee Cb$, and (iv) $C0 = 0$. We call C a *closure operator* on B.

To give an example of a closure algebra, let X be a qoset and let $\mathcal{P}(X)$ denote the powerset of X. It is easy to check that $\downarrow$ is a closure operator on $\mathcal{P}(X)$. Hence, $(\mathcal{P}(X), \downarrow)$ is a closure algebra. We call $(\mathcal{P}(X), \downarrow)$ the *closure algebra over the qoset* X. More generally, if X is a topological space, then it is routine to verify that $(\mathcal{P}(X), ^{-})$ is a closure algebra. We call $(\mathcal{P}(X), ^{-})$ the *closure algebra over the topological space* X.

Suppose X and Y are topological spaces and $f : X \to Y$ is an open map. Then it is easy to verify that for $A \subseteq Y$ we have $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$. Therefore, $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a closure algebra homomorphism. Moreover, if f is onto, then f^{-1} is one-to-one, and hence $(\mathcal{P}(Y), ^{-})$ is isomorphic to a subalgebra of $(\mathcal{P}(X), ^{-})$.

Theorem 1. (a) *Every closure algebra can be represented as a subalgebra of the closure algebra over a topological space. In fact, every closure algebra can be represented as a subalgebra of the closure algebra over an Alexandroff space, or equivalently, over a qoset.*

(b) *If a closure algebra is finite, then it is isomorphic to the closure algebra over a finite space, or equivalently, over a finite qoset.*

(c) *A finite closure algebra is subdirectly irreducible iff it is isomorphic to the closure algebra over a finite well-connected space, or equivalently, over a finite rooted qoset.*

(d) **S4** *is complete with respect to finite subdirectly irreducible closure algebras. Hence,* **S4** *is complete with respect to the closure algebras over finite well-connected spaces, or equivalently, over finite rooted qosets.*

Proof. In the light of the above correspondence between Alexandroff spaces and qosets, (a) follows from [8, Theorem 2.4] and [6, Theorem 3.14]; (b) follows from [3, Lemma 1]; (c) follows from [4, the paragraph after the Theorem of Duality]; and finally, (d) follows from [8, Theorem 4.16]. □

Let $\mathcal{L}(\forall)$ denote the enrichment of $\mathcal{L}$ by the universal modality $\forall$. As usual, the existential modality $\exists$ is the abbreviation of $\neg\forall\neg$. We recall that Bennett's logic **S4** + **S5** is the least set of formulae of $\mathcal{L}(\forall)$ containing the $\Box$-axioms for **S4**, the $\forall$-axioms for **S5** (that is $\forall$-axioms for **S4** plus the axiom $\exists\varphi \to \forall\exists\varphi$), the bridge axiom $\forall\varphi \to \Box\varphi$, and closed under modus ponens, substitution, $\Box$-necessitation, and $\forall$-necessitation $(\varphi/\forall\varphi)$.

Algebraic models of **S4** + **S5** are the triples $(B, C, \exists)$, where (i) (B, C) is a closure algebra, (ii) $(B, \exists)$ is a *monadic algebra* (that is $(B, \exists)$ is a closure algebra satisfying the identity $\exists - \exists a = -\exists a$), and (iii) $Ca \leq \exists a$. We call $(B, C, \exists)$ an (**S4** + **S5**)-*algebra*.

Examples of (**S4** + **S5**)-algebras can be obtained from the closure algebras over topological spaces. Let X be a topological space. We define $\exists$ on $\mathcal{P}(X)$ by setting

$$\exists A = \begin{cases} \emptyset, & \text{if } A = \emptyset \\ X, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{P}(X), ^{-}, \exists)$ is an (**S4** + **S5**)-algebra, called the (**S4** + **S5**)-*algebra over the topological space* X. In particular, if X is a qoset, then $(\mathcal{P}(X), \downarrow, \exists)$ is an (**S4** + **S5**)-algebra, called the (**S4** + **S5**)-*algebra over the qoset* X.

Theorem 2. (a) *Every* $(\mathbf{S4}+\mathbf{S5})$*-algebra over a topological space is simple (has no proper congruences).*
(b) *Every simple* $(\mathbf{S4}+\mathbf{S5})$*-algebra can be represented as a subalgebra of the* $(\mathbf{S4}+\mathbf{S5})$*-algebra over some (Alexandroff) space.*
(c) *If a simple* $(\mathbf{S4}+\mathbf{S5})$*-algebra is finite, then it is isomorphic to the* $(\mathbf{S4}+\mathbf{S5})$*-algebra over a finite space, or equivalently, over a finite qoset.*
(d) $\mathbf{S4}+\mathbf{S5}$ *is complete with respect to finite simple* $(\mathbf{S4}+\mathbf{S5})$*-algebras. Hence,* $\mathbf{S4}+\mathbf{S5}$ *is complete with respect to the* $(\mathbf{S4}+\mathbf{S5})$*-algebras over finite topological spaces, or equivalently, over finite qosets.*

Proof. For (a) see [5, Lemma 3.1]. For (b) observe that a $(\mathbf{S4}+\mathbf{S5})$-algebra $(B, C, \exists)$ is simple iff for every $a \in B$ we have $a \neq 0$ implies $\exists a = 1$. Now apply Theorem 1(a). (c) follows from (b) and Theorem 1(b). For (d) see [13, Theorem 7] or [5, Theorem 5.9]. □

It was proved in [13, Lemma 8] that the connectedness axiom

$$\mathbf{C} = \forall(\Diamond\varphi \to \Box\varphi) \to (\forall\varphi \vee \forall\neg\varphi)$$

is valid in the $(\mathbf{S4}+\mathbf{S5})$-algebra over a topological space X iff X is connected. In particular, $\mathbf{C}$ is valid in the $(\mathbf{S4}+\mathbf{S5})$-algebra over a qoset X iff X is a connected component. Let $\mathbf{S4}+\mathbf{S5}+\mathbf{C}$ denote the normal extension of $\mathbf{S4}+\mathbf{S5}$ by the connectedness axiom. We call an $(\mathbf{S4}+\mathbf{S5})$-algebra $(B, C, \exists)$ a $(\mathbf{S4}+\mathbf{S5}+\mathbf{C})$*-algebra* if the connectedness axiom is valid in $(B, C, \exists)$.

Theorem 3. $\mathbf{S4}+\mathbf{S5}+\mathbf{C}$ *is complete with respect to finite simple* $(\mathbf{S4}+\mathbf{S5}+\mathbf{C})$*-algebras. Hence,* $\mathbf{S4}+\mathbf{S5}+\mathbf{C}$ *is complete with respect to the* $(\mathbf{S4}+\mathbf{S5}+\mathbf{C})$*-algebras over finite connected spaces, or equivalently, over finite connected components.*

Proof. See [13, Theorem 10]. □

3. Completeness of S4

We recall that a subset A of $\mathbb{R}$ is said to be *convex* if $x, y \in A$ and $x \leq z \leq y$ imply that $z \in A$. We denote by $C(\mathbb{R})$ the set of convex subsets of $\mathbb{R}$, and by $C^\infty(\mathbb{R})$ the set of countable unions of convex subsets of $\mathbb{R}$. We also let $B(C^\infty(\mathbb{R}))$ denote the Boolean algebra generated by $C^\infty(\mathbb{R})$. It is obvious that every open interval of $\mathbb{R}$ belongs to $C(\mathbb{R})$. Now since every open subset of $\mathbb{R}$ is a countable union of open intervals of $\mathbb{R}$, it follows that every open subset of $\mathbb{R}$, and hence every closed subset of $\mathbb{R}$ belongs to $B(C^\infty(\mathbb{R}))$. Therefore, $(B(C^\infty(\mathbb{R})), ^-)$ is a closure algebra. In fact, $(B(C^\infty(\mathbb{R})), ^-)$ is a proper subalgebra of $(\mathcal{P}(\mathbb{R}), ^-)$. Our goal is to show that $\mathbf{S4}$ is complete with respect to $(B(C^\infty(\mathbb{R})), ^-)$. For this, as follows from Theorem 1, it is sufficient to show that every closure algebra over a finite rooted qoset is isomorphic to a subalgebra of $(B(C^\infty(\mathbb{R})), ^-)$.

Suppose X is a finite poset. We call $Y \subseteq X$ a *chain* if for every $x, y \in Y$ we have $x \leq y$ or $y \leq x$. For $x \in X$ let $d(x)$ be the number of elements of a maximal chain with the root x; we call $d(x)$ the *depth* of x. Let also $d(X) = \sup\{d(x) : x \in X\}$; we call $d(X)$ the *depth* of X. For $x, y \in X$ let $x < y$ mean that $x \leq y$ and $x \neq y$. We call y an *immediate successor* of x if $x < y$ and there is no z such that $x < z < y$. For $x \in X$ let $b(x)$ be the number of immediate successors of x; we call $b(x)$ the *branching* of x. Let also

$b(X) = \sup\{b(x) : x \in X\}$; we call $b(X)$ the *branching* of X. A finite poset X is called a *tree* if $\downarrow x$ is a chain for every $x \in X$; if in the tree X we have $b(x) = n$ for every $x \in X$, then we call X an *n-tree*.

Lemma 4. (a) *Every finite rooted poset is a p-morphic image of a finite tree.*

(b) *Every tree of branching n and depth m is a p-morphic image of the n-tree of depth m.*

(c) *For every finite rooted poset X there exists n such that X is a p-morphic image of a finite n-tree.*

Proof. For (a) see [7, Proposition 2]; (b) follows from [7, Theorem 1]; finally, (c) follows from (a) and (b). □

We call a finite qoset X *q-regular* if every cluster of X consists of exactly q elements. We define an equivalence relation $\sim$ on X by putting $x \sim y$ iff $C[x] = C[y]$. Let $X/\!\sim$ denote the quotient of X under $\sim$, where $[x] \leq_\sim [y]$ if there exist $x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$. Obviously $X/\!\sim$ is a finite poset, called the *skeleton* of X. We call X a *quasi-tree* if $X/\!\sim$ is a tree; we call X a *quasi-n-tree* if $X/\!\sim$ is an n-tree; finally, we call X a *quasi-(q, n)-tree* if X is a q-regular quasi-n-tree. The following lemma is an easy generalization of Lemma 4 to qosets.

Lemma 5. *For every finite rooted qoset X there exist q, n such that X is a p-morphic image of a finite quasi-(q, n)-tree.*

Proof (Sketch). Let $q = \sup\{|C[x]| : x \in X\}$. Then replacing every cluster of X by a q-element cluster, we get a new q-regular qoset Y. Obviously X is a p-morphic image of Y and $X/\!\sim$ is isomorphic to $Y/\!\sim$. From the previous lemma we know that there exist an n-tree T_n and a p-morphism f from T_n onto $Y/\!\sim$. We denote by $T_{q,n}$ the quasi-tree obtained from T_n by replacing every node t of T_n by a q-element cluster $[t] = \{t_1, \dots, t_q\}$. Obviously $T_{q,n}$ is a finite quasi-(q, n)-tree and T_n is (isomorphic to) $T_{q,n}/\!\sim$. Suppose $[y] = \{y_1, \dots, y_q\}$ is an element of $Y/\!\sim$ and $[t] = \{t_1, \dots, t_q\}$ is an element of $T_{q,n}/\!\sim\, = T_n$. We define $h : T_{q,n} \to Y$ by putting $h(t_i) = y_i$ if $f([t]) = [y]$, $t_i \in [t]$, and $y_i \in [y]$ for $1 \leq i \leq q$. Since $[h(t_i)] = f([t])$ and f is an onto p-morphism, so is h. So Y is a p-morphic image of $T_{q,n}$, and since X is a p-morphic image of Y, it is also a p-morphic image of $T_{q,n}$. □

Corollary 6. **S4** *is complete with respect to the closure algebras over finite quasi-trees.*

Proof. It follows from Theorem 1(d) that **S4** is complete with respect to the closure algebras over finite rooted qosets. From Lemma 5 it follows that the closure algebra over a finite rooted qoset is isomorphic to a subalgebra of the closure algebra over some finite quasi-tree. Thus, **S4** is complete with respect to the closure algebras over finite quasi-trees. □

Now we are in a position to show that finite rooted qosets are open images of $\mathbb{R}$. We first show that every finite rooted poset is an open image of $\mathbb{R}$, and then extend this result to finite qosets. Let us start by showing that the n-tree T of depth 2 shown in Fig. 1 is an open image of any bounded interval $I \subseteq \mathbb{R}$.

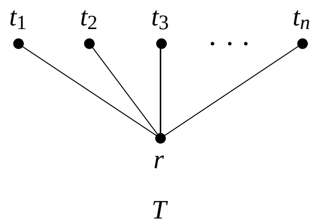

Fig. 1. An n-tree of depth 2.

Suppose $a, b \in \mathbb{R}$, $a < b$, $I = (a, b)$, $I = [a, b)$, $I = (a, b]$, or $I = [a, b]$. We recall that the Cantor set $\mathcal{C}$ is constructed inside I by taking out open intervals from I infinitely many times. More precisely, in step 1 of the construction the open interval

$$I_1^1 = \left(a + \frac{b-a}{3}, a + \frac{2(b-a)}{3}\right)$$

is taken out. We denote the remaining closed intervals by J_1^1 and J_2^1. In step 2 the open intervals

$$I_1^2 = \left(a + \frac{b-a}{3^2}, a + \frac{2(b-a)}{3^2}\right) \quad \text{and} \quad I_2^2 = \left(a + \frac{7(b-a)}{3^2}, a + \frac{8(b-a)}{3^2}\right)$$

are taken out. We denote the remaining closed intervals by J_1^2, J_2^2, J_3^2, and J_4^2. In general, in step m the open intervals $I_1^m, \dots, I_{2^{m-1}}^m$ are taken out, and the closed intervals $J_1^m, \dots, J_{2^m}^m$ remain. We will use the construction of $\mathcal{C}$ to obtain T as an open image of I.

Lemma 7. *T is an open image of I.*

Proof. Define $f_I^T : I \to T$ by putting

$$f_I^T(x) = \begin{cases} t_k, & \text{if } x \in \bigcup_{m \equiv k (\text{mod } n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \\ r, & \text{otherwise} \end{cases}.$$

Obviously, f_I^T is a well-defined onto map. Moreover,

$$(f_I^T)^{-1}(t_k) = \bigcup_{m \equiv k (\text{mod } n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \qquad \text{and} \qquad (f_I^T)^{-1}(r) = \mathcal{C}.$$

Let us show that f_I^T is open. Since $\{\emptyset, \{t_1\}, \dots, \{t_n\}, T\}$ is a family of basic open subsets of T, continuity of f_I^T is obvious. Suppose U is an open interval in I. If $U \cap \mathcal{C} = \emptyset$, then $f_I^T(U) \subseteq \{t_1, \dots, t_n\}$. Thus, $f_I^T(U)$ is open. If $U \cap \mathcal{C} \neq \emptyset$, then there exists $c \in U \cap \mathcal{C}$. Since $c \in \mathcal{C}$ we have $f_I^T(c) = r$. From $c \in U$ it follows that there is $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. We pick m so that $\frac{b-a}{3^m} < \varepsilon$. As $c \in \mathcal{C}$, there is $k \in \{1, \dots, 2^m\}$ such that $c \in J_k^m$. Moreover, since the length of J_k^m is equal to $\frac{b-a}{3^m}$, we have that $J_k^m \subseteq U$. Therefore, U contains the points removed from J_k^m in the subsequent iterations in the

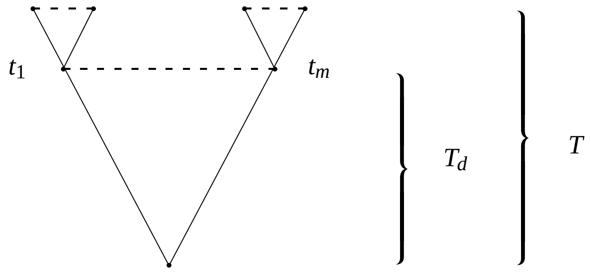

Fig. 2. T and T_d.

construction of $\mathcal{C}$. Thus, $f_I^T(U) \supseteq \{t_1, \ldots, t_n\}$ and $f_I^T(U) = T$. Hence, $f_I^T(U)$ is open for any open interval U of I. It follows that f_I^T is an onto open map. □

Theorem 8. *Every finite n-tree is an open image of I.*

Proof. For an arbitrary finite n-tree T we define a map $f_I : I \to T$ by induction on the depth of T. If the depth of T is 1, then T is a 1-tree consisting of a single element t, and for every $x \in I$ we set $f_I(x) = t$. Then it is obvious that f_I is onto and open. If the depth of T is 2, then for every $x \in I$ we define $f_I(x) = f_I^T(x)$. Then the previous lemma guarantees that f_I is onto and open. Now suppose the depth of T is $d + 1$, $d \geq 2$. Let $t_1, \ldots, t_m$ $(m = n^d)$ be the elements of T of depth 2, and let T_d be the subtree of T of all elements of T of depth ≥ 2 (see Fig. 2).

We note that for each $k \in \{1, \ldots, m\}$ the upset $\uparrow t_k$ is isomorphic to the n-tree of depth 2, and that T_d is the n-tree of depth d. So by the induction hypothesis there exists an onto open map $f_I^d : I \to T_d$. We use f_I^d to define $f_I : I \to T$ as follows. For each $k \in \{1, \ldots, m\}$ and $x \in (f_I^d)^{-1}(t_k)$ let I_x denote the connected component of $(f_I^d)^{-1}(t_k)$ containing x. We set

$$f_I(x) = \begin{cases} f_I^d(x), & \text{if } f_I^d(x) \notin \{t_1, \ldots, t_m\} \\ f_{I_x}^{\uparrow t_k}(x), & \text{if } f_I^d(x) = t_k. \end{cases}$$

It is clear that f_I is a well-defined onto map. To show that f_I is continuous observe that for $t \in T - T_d$ there is a unique t_k such that $t_k < t$. Hence, we have

$$f_I^{-1}(t) = \bigcup \{(f_{I'}^{\uparrow t_k})^{-1}(t) : I' \text{ is a connected component of } (f_I^d)^{-1}(t_k)\}.$$

Also for $t \in T_d$ we have

$$f_I^{-1}(\uparrow_T t) = (f_I^d)^{-1}(\uparrow_{T_d} t).$$

Now since the family $\{\emptyset\} \cup \{\{t\} : t \in T - T_d\} \cup \{\uparrow_T t : t \in T_d\}$ forms a basis for T, we have that f_I is continuous.

To show that f_I is open, let $U = (c, d)$ be an open interval in I. If $U \subseteq I'$ where I' is a connected component of $(f_I^d)^{-1}(t_k)$ for some k, then $f_I(U) = f_{I'}^{\uparrow t_k}(U)$. Therefore, $f_I(U)$ is open by the previous lemma. Assume $U \not\subseteq I'$ for any k and I'. We want to show that $f_I(U) = \uparrow f_I^d(U)$. If $t \in T - \uparrow\{t_1, \ldots, t_m\}$, then $f_I^{-1}(t) = (f_I^d)^{-1}(t)$, and thus $t \in f_I(U)$

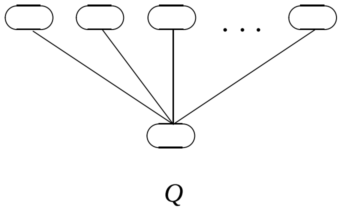

Fig. 3. A quasi-(q, n)-tree of depth 2.

iff $t \in f_I^d(U)$. So we can assume that $t \in \uparrow t_k$ for some k. Then if $t \in f_I(U)$, there is $x \in U$ with $f_I(x) = t$. Hence, by the definition of f_I, there exists a connected component I' of $(f_I^d)^{-1}(t_k)$ with $x \in I'$ and $f_I(x) = f_{I'}^{\uparrow t_k}(x)$. Therefore, $x \in U \cap (f_I^d)^{-1}(t_k)$, which implies that $t_k \in f_I^d(U)$. Hence, $t \in \uparrow t_k \subseteq \uparrow f_I^d(U)$. Conversely, if $t \in \uparrow f_I^d(U)$, then there exist $k \in \{1, \ldots, m\}$ and $x \in U$ with $f_I^d(x) = t_k \leq t$. Hence, $x \in (f_I^d)^{-1}(t_k)$, and there is a connected component $I' = (p, q)$ of $(f_I^d)^{-1}(t_k)$ containing x. Since $U \cap I' \neq \emptyset$ and by assumption $U \not\subseteq I'$, we have that $U \cap I'$ is either (p, d) or (c, q). As both (p, d) and (c, q) must intersect the Cantor set constructed in I' and $f_{I'}^{\uparrow t_k}$ is open, we have $f_I(U) \supseteq f_I(U \cap I') = f_{I'}^{\uparrow t_k}(U \cap I') = \uparrow t_k$. It follows that $t \in \uparrow t_k \subseteq f_I(U)$. Therefore, $f_I(U) = \uparrow f_I^d(U)$, and so $f_I(U)$ is open. Thus, f_I is an onto open map, implying that T is an open image of I. $\square$

Corollary 9. *Every finite rooted poset, or equivalently, every finite well-connected T_0-space is an open image of $\mathbb{R}$.*

Proof. It follows from Lemma 4 and Theorem 8 that every finite rooted poset is an open image of any bounded interval $I \subseteq \mathbb{R}$. In particular, if I is open, then I is homeomorphic to $\mathbb{R}$, and so the corollary follows. $\square$

Remark 10. It follows from Corollary 9 that the Heyting algebra of upsets of a finite rooted poset is isomorphic to a subalgebra of the Heyting algebra $\mathcal{O}(\mathbb{R})$ of open subsets of $\mathbb{R}$. Hence, every finite subdirectly irreducible Heyting algebra is isomorphic to a subalgebra of $\mathcal{O}(\mathbb{R})$. This together with the finite model property of the intuitionistic propositional logic **Int** gives a new proof of completeness of **Int** with respect to $\mathcal{O}(\mathbb{R})$, a fact first established by Tarski [14] back in 1938. Now, applying the Blok–Esakia theorem, we obtain that the Grzegorczyk modal system $\mathbf{Grz} = \mathbf{S4} + \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$ is complete with respect to the Boolean closure $B(\mathcal{O}(\mathbb{R}))$ of $\mathcal{O}(\mathbb{R})$.

We are now in a position to expand on Corollary 9 and show that finite rooted qosets are open images of $\mathbb{R}$. We start by showing that the quasi-(q, n)-tree Q of depth 2 shown in Fig. 3 is an open image of I.

Lemma 11. *If X has a countable basis and every countable subset of X is boundary, then for any natural number n there exist disjoint dense boundary subsets $A_1, \ldots, A_n$ of X such that $X = \bigcup_{i=1}^n A_i$.*

Proof. Suppose $\{B_i\}_{i=1}^{\infty}$ is a countable basis of X. Since every countable subset of X is boundary, each B_i is uncountable. We pick from each B_i a point x_i^1 and set $A_1 = \{x_i^1\}_{i=1}^{\infty}$. Since A_1 is countable, each $B_i - A_1$ is uncountable. So we pick from each $B_i - A_1$ a point x_i^2 and set $A_2 = \{x_i^2\}_{i=1}^{\infty}$. We repeat the same construction for each $B_i - (A_1 \cup A_2)$ to obtain A_3. After repeating the construction $n-1$ times we obtain $n-1$ many sets $A_1, \ldots, A_{n-1}$. Finally, we set $A_n = X - \bigcup_{i=1}^{n-1} A_i$. It is clear that different A_i's are disjoint from each other and that $X = \bigcup_{i=1}^{n} A_i$. Moreover, each A_i contains at least one point from every basic open set. Hence, each A_i is dense. Furthermore, no basic open set is a subset of any A_i. Therefore, every A_i is boundary. □

Lemma 12. *Q is an open image of I.*

Proof. We denote the least cluster of Q by r and its elements by $r_1, \ldots, r_q$. Also for $1 \le i \le n$ we denote the i-th maximal cluster of Q by t^i and its elements by $t_1^i, \ldots, t_q^i$. Since the Cantor set $\mathcal{C}$ satisfies the conditions of Lemma 11, it can be divided into q-many disjoint dense boundary subsets $\mathcal{C}_1, \ldots, \mathcal{C}_q$. Also each I_p^m ($1 \le p \le 2^{m-1}$, $m \in \omega$) satisfies the conditions of Lemma 11, and so each I_p^m can be divided into q-many disjoint dense boundary subsets $(I_p^m)^1, \ldots, (I_p^m)^q$. Suppose $1 \le k \le q$. We define $f_I^Q : I \to Q$ by putting

$$f_I^Q(x) = \begin{cases} t_k^i, & \text{if } x \in \bigcup_{m \equiv i (\mathrm{mod}\ n)} \bigcup_{p=1}^{2^{m-1}} (I_p^m)^k \\ r_k, & \text{if } x \in \mathcal{C}_k \end{cases}.$$

It is clear that f_I^Q is a well-defined onto map. Similar to Lemma 7 we have

$$(f_I^Q)^{-1}(t^i) = \bigcup_{m \equiv i (\mathrm{mod}\ n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \qquad \text{and} \qquad (f_I^Q)^{-1}(r) = \mathcal{C}.$$

Hence, f_I^Q is continuous. To show that f_I^Q is open let U be an open interval in I. If $U \cap \mathcal{C} = \emptyset$, then $f_I^Q(U) \subseteq \bigcup_{i=1}^{n} t^i$. Moreover, since $(I_p^m)^1, \ldots, (I_p^m)^q$ partition I_p^m into q-many disjoint dense boundary subsets, $U \cap I_p^m \neq \emptyset$ implies $U \cap (I_p^m)^k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, if $f_I^Q(U)$ contains an element of a cluster t^i, it contains the whole cluster. Thus, $f_I^Q(U)$ is open. Now suppose $U \cap \mathcal{C} \neq \emptyset$. Since $\mathcal{C}_1, \ldots, \mathcal{C}_q$ partition $\mathcal{C}$ into q-many disjoint dense boundary subsets, $U \cap \mathcal{C}_k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, $r \subseteq f_I^Q(U)$. Moreover, the same argument as in the proof of Lemma 7 guarantees that every point greater than points in r also belongs to $f_I^Q(U)$. Thus $f_I^Q(U) = Q$, implying that f_I^Q is an onto open map. □

Theorem 13. *Every finite quasi-(q, n)-tree is an open image of I.*

Proof. This follows along the same lines as the proof of Theorem 8 but is based on Lemma 12 instead of Lemma 7. □

Corollary 14. *Every finite rooted qoset, or equivalently, every finite well-connected space is an open image of $\mathbb{R}$.*

Proof. This follows along the same lines as the proof of Corollary 9 but is based on Lemma 5 and Theorem 13 instead of Lemma 4 and Theorem 8. □

Theorem 15. **S4** *is complete with respect to* $(B(C^\infty(\mathbb{R})), ^-)$.

Proof. It is sufficient to show that the closure algebra over a quasi-(q, n)-tree is isomorphic to a subalgebra of $(B(C^\infty(\mathbb{R})), ^-)$. So let X be a quasi-(q, n)-tree and I be a bounded interval of $\mathbb{R}$. We denote by $\mathcal{C}$ the Cantor set constructed inside I, and by $\mathcal{C}_1, \ldots, \mathcal{C}_q$ disjoint dense boundary subsets of $\mathcal{C}$ constructed in Lemma 11. By Theorem 13 there exists an onto open map $f_I : I \to X$. We show that for every $x \in X$ we have $(f_I)^{-1}(x) \in B(C^\infty(I))$. If x is a quasi-minimal point of X, then by Lemma 12 $(f_I)^{-1}(x) = \mathcal{C}_k$ for some $k \in \{1, \ldots, q\}$. From the proof of Lemma 11 it follows that either $\mathcal{C}_k$ or $\mathcal{C} - \mathcal{C}_k$ is a countable subset of I. In either case we have $(f_I)^{-1}(x) \in B(C^\infty(I))$. Now suppose x is neither a quasi-minimal nor a quasi-maximal point of X. Then by the proof of Theorem 13, which follows along the same lines as the proof of Theorem 8, $(f_I)^{-1}(x)$ is a countable union of the sets $\mathcal{C}_k^{I'}$, where each $\mathcal{C}_k^{I'}$ is a dense boundary subset of the Cantor set $\mathcal{C}^{I'}$ constructed inside some open interval I' of I. Let U denote the (countable) union of these open intervals. Then by Lemma 11 $(f_I)^{-1}(x)$ or $U - (f_I)^{-1}(x)$ is countable. Thus, $(f_I)^{-1}(x) \in B(C^\infty(I))$. Finally, if x is a quasi-maximal point of X, then $(f_I)^{-1}(x) = \bigcup_{m \equiv i (\text{mod } n)} \bigcup_{p=1}^{2^{m-1}} (I_p^m)^k$ for some $k \in \{1, \ldots, q\}$, where each $(I_p^m)^k$ is a dense boundary subset of the interval I_p^m constructed inside some open interval of I. Let U denote the (countable) union of these open intervals. Then the same argument as above guarantees that $(f_I)^{-1}(x)$ or $U - (f_I)^{-1}(x)$ is countable. Therefore, $(f_I)^{-1}(x) \in B(C^\infty(I))$. Thus, the closure algebra over a quasi-(q, n)-tree is isomorphic to a subalgebra of $(B(C^\infty(I)), ^-)$. Now if I is an open interval, then I is homeomorphic to $\mathbb{R}$. Hence, the closure algebra over a quasi-(q, n)-tree is isomorphic to a subalgebra of $(B(C^\infty(\mathbb{R})), ^-)$, and so **S4** is complete with respect to $(B(C^\infty(\mathbb{R})), ^-)$. □

4. Completeness of S4 + S5 + C

In this section we show that **S4** + **S5** + **C** is complete with respect to the algebra $(B(C^\infty(\mathbb{R})), ^-, \exists)$. For this, by Theorem 3, it is sufficient to construct an open map from $\mathbb{R}$ onto every finite connected component X such that for every $x \in X$ we have $f^{-1}(x) \in B(C^\infty(\mathbb{R}))$.

Suppose $T_1, \ldots, T_n$ are finite trees (of branching ≥ 2). Let t_i^l and t_i^r denote two distinct maximal nodes of T_i. Consider the disjoint union $\bigsqcup_{i=1}^n T_i$, and identify t_{i-1}^r with t_i^l and t_i^r with t_{i+1}^l. We call this construction the *tree sum* of $T_1, \ldots, T_n$ and denote it by $\bigoplus_{i=1}^n T_i$ (see Fig. 4).

We can generalize this construction to quasi-trees. Suppose $Q_1, \ldots, Q_n$ are finite q-regular quasi-trees (of branching ≥ 2). Let C_i^l and C_i^r denote two distinct maximal clusters of Q_i. Consider the disjoint union $\bigsqcup_{i=1}^n Q_i$, and identify C_{i-1}^r with C_i^l and C_i^r with C_{i+1}^l. We call this construction the *regular quasi-tree sum* of $Q_1, \ldots, Q_n$ and denote it by $\bigoplus_{i=1}^n Q_i$.

Lemma 16 (Compare with [13, Lemma 13]).

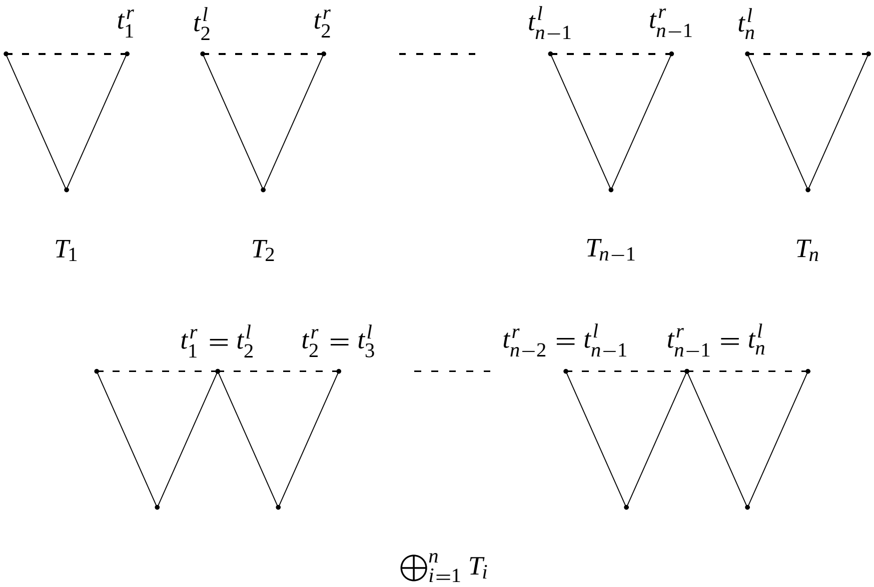

Fig. 4. Construction of $\bigoplus_{i=1}^n T_i$ from $T_1, \ldots, T_n$.

(a) *For every finite partially ordered connected component X there exist trees $T_1, \ldots, T_n$ such that X is a p-morphic image of $\bigoplus_{i=1}^n T_i$.*
(b) *For every finite connected component X there exist q-regular quasi-trees $Q_1, \ldots, Q_n$ such that X is a p-morphic image of $\bigoplus_{i=1}^n Q_i$.*

Proof. (a) follows from (b) and the fact that the regular quasi-tree sum of trees is in fact their tree sum.

(b) Suppose X is a finite connected component. Let $C_1, \ldots, C_n$ denote minimal clusters of X. Consider $(\uparrow C_1, \leq_1), \ldots, (\uparrow C_n, \leq_n)$, where $\leq_i$ is the restriction of $\leq$ to $\uparrow C_i$. Obviously each $(\uparrow C_i, \leq_i)$ is a finite rooted qoset and $\bigcup_{i=1}^n C_i = X$. As follows from Lemma 5, for each $(\uparrow C_i, \leq_i)$ there exist q_i, m_i such that $(\uparrow C_i, \leq_i)$ is a p-morphic image of a finite quasi-(q_i, m_i)-tree. Let $q = \sup\{q_1, \ldots, q_n\}$, and consider quasi-$(q, m_i)$-trees $Q_1, \ldots, Q_n$. Obviously for each i there exists a p-morphism f_i from Q_i onto $(\uparrow C_i, \leq_i)$. Also note that for each i there exists a maximal cluster C of X such that C is a subset of both $\uparrow C_{i-1}$ and $\uparrow C_i$. Since f_{i-1} is a p-morphism, there exists a maximal cluster D_{i-1}^r of Q_{i-1} such that $f_{i-1}(D_{i-1}^r) = C$. Similarly there exists a maximal cluster D_i^l of Q_i such that $f_i(D_i^l) = C$. We form $\bigoplus_{i=1}^n Q_i$ by identifying D_{i-1}^r with D_i^l and D_i^r with D_{i+1}^l. Now define $f : \bigoplus_{i=1}^n Q_i \to X$ by putting $f(t) = f_i(t)$ for $t \in Q_i$. It is routine to check that f is well defined and that it is an onto p-morphism. $\square$

Theorem 17. *The tree sum of finitely many finite trees is an open image of $\mathbb{R}$.*

Proof. Suppose $T_1, \ldots, T_n$ are finite trees. Consider $\bigoplus_{k=1}^n T_k$. For $2 \leq k \leq n-1$ let t_k^l and t_k^r denote the maximal nodes of T_k which got identified with the corresponding nodes

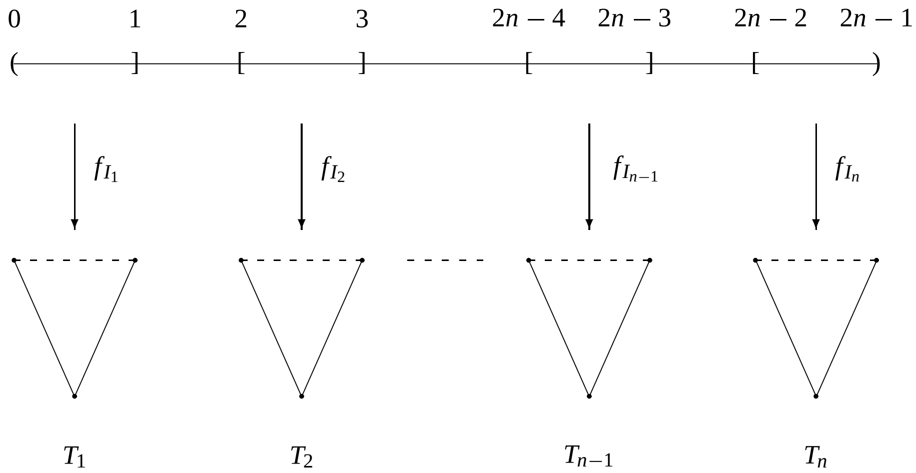

Fig. 5. The maps f_{I_k}.

t^r_{k-1} of T_{k-1} and t^l_{k+1} of T_{k+1}, respectively. Also let $I_1 = (0, 1]$, $I_k = [2k-2, 2k-1]$ for $k \in \{2, \ldots, n-1\}$, and $I_n = [2n-2, 2n-1)$. From Theorem 8 it follows that for each I_k there exists an onto open map $f_{I_k} : I_k \to T_k$ (see Fig. 5).

We define $f : (0, 2n-1) \to \bigoplus_{k=1}^{n} T_k$ by putting

$$f(x) = \begin{cases} f_{I_k}(x), & \text{if } x \in I_k \\ t^r_k, & \text{if } x \in (2k-1, 2k) \\ f_{I_n}(x), & \text{if } x \in I_n \end{cases}$$

where $k \in \{1, \ldots, n-1\}$. It is obvious that f is a well-defined onto map. For $t \in T_k$ observe that if $t^l_k, t^r_k \notin {\uparrow}t$, then

$$f^{-1}({\uparrow}t) = f^{-1}_{I_k}({\uparrow}t),$$

if $t^l_k \in {\uparrow}t$ and $t^r_k \notin {\uparrow}t$, then

$$f^{-1}({\uparrow}t) = f^{-1}_{I_{k-1}}(t^r_{k-1}) \cup (2k-3, 2k-2) \cup f^{-1}_{I_k}({\uparrow}t),$$

if $t^l_k \notin {\uparrow}t$ and $t^r_k \in {\uparrow}t$, then

$$f^{-1}({\uparrow}t) = f^{-1}_{I_k}({\uparrow}t) \cup (2k-1, 2k) \cup f^{-1}_{I_{k+1}}(t^l_{k+1}),$$

and finally, if $t^l_k, t^r_k \in {\uparrow}t$, then

$$f^{-1}({\uparrow}t) = f^{-1}_{I_{k-1}}(t^r_{k-1}) \cup (2k-3, 2k-2) \cup f^{-1}_{I_k}({\uparrow}t) \cup (2k-1, 2k) \cup f^{-1}_{I_{k+1}}(t^l_{k+1}).$$

Hence, f is continuous. Moreover, for an open interval $U \subseteq (0, 2n-1)$, if $U \subseteq I_k$, then $f(U) = f_{I_k}(U)$; and if $U \subseteq (2k-1, 2k)$, then $f(U) = \{t^r_k\}$. In either case $f(U)$ is open in $\bigoplus_{k=1}^{n} T_k$. Now every open interval $U \subseteq (0, 2n-1)$ is the union $U = U_1 \cup \ldots \cup U_{2n-1}$, where $U_{2k} = U \cap (2k-1, 2k)$ for $k = 1, \ldots, n-1$, and $U_{2k+1} = U \cap I_{k+1}$ for

$k = 0, \dots, n-1$. Thus, $f(U) = f(U_1) \cup \dots \cup f(U_{2n-1})$, and so $f(U)$ is an open set in $\bigoplus_{k=1}^n T_k$. Hence, f is an onto open map, implying that $\bigoplus_{k=1}^n T_k$ is an open image of $(0, 2n-1)$. Since $(0, 2n-1)$ is homeomorphic to $\mathbb{R}$, we obtain that $\bigoplus_{k=1}^n T_k$ is an open image of $\mathbb{R}$. □

Corollary 18. *A finite T_0-space is an open image of $\mathbb{R}$ iff it is connected.*

Proof. Since finite connected T_0-spaces correspond to finite connected partially ordered components, it follows from Lemma 16 and Theorem 17 that every finite connected T_0-space is an open image of $\mathbb{R}$. Conversely, since $\mathbb{R}$ is connected and open (even continuous) images of connected spaces are connected, finite T_0 images of $\mathbb{R}$ are connected. □

Theorem 19. *The regular quasi-tree sum of finitely many finite q-regular quasi-trees is an open image of $\mathbb{R}$.*

Proof. This follows along the same lines as the proof of Theorem 17 but is based on Theorem 13 instead of Theorem 8. In addition, according to Lemma 11, for $k = 1, \dots, n-1$ we divide each interval $(2k-1, 2k)$ into q-many disjoint dense boundary subsets $A_1^k, \dots, A_q^k$ and define $f : (0, 2n-1) \to \bigoplus_{k=1}^n Q_k$ by putting

$$f(x) = \begin{cases} f_{I_k}(x), & \text{if } x \in I_k \\ (t_k^r)_i, & \text{if } x \in A_i^k \\ f_{I_n}(x), & \text{if } x \in I_n \end{cases}$$

where $(t_k^r)_i$ is the i-th element of C_k^r and $k \in \{1, \dots, n-1\}$. As a result we obtain that $\bigoplus_{k=1}^n Q_k$ is an open image of $(0, 2n-1)$, and so $\bigoplus_{k=1}^n Q_k$ is an open image of $\mathbb{R}$. □

Corollary 20. *A finite topological space is an open image of $\mathbb{R}$ iff it is connected.*

Proof. This follows along the same lines as the proof of Corollary 18 but is based on Theorem 19 instead of Theorem 17. □

Theorem 21. **S4 + S5 + C** *is complete with respect to* $(B(C^\infty(\mathbb{R})), ^-, \exists)$.

Proof. Suppose $Q_1, \dots, Q_n$ are arbitrary q-regular quasi-trees. It is sufficient to show that the (**S4 + S5 + C**)-algebra over the regular quasi-tree sum $\bigoplus_{k=1}^n Q_k$ is isomorphic to a subalgebra of $(B(C^\infty(\mathbb{R})), ^-, \exists)$. The proof of Theorem 15 implies that for each Q_k there exists $I_k = [2k-2, 2k-1]$ and an onto open map $f_k : I_k \to Q_k$ such that for every $t \in Q_k$ we have $f_k^{-1}(t) \in B(C^\infty(I_k))$. It follows from the proof of Theorem 19 that there exists an onto open map $f : (0, 2n-1) \to \bigoplus_{k=1}^n Q_k$. If $t \in Q_k$ does not belong to either C_k^l or C_k^r, then $f^{-1}(t) = f_k^{-1}(t)$, and so $f^{-1}(t) \in B(C^\infty(0, 2n-1))$. If $t \in C_k^l$, then $f^{-1}(t)$ is the union of $f_k^{-1}(t) \cup f_{k-1}^{-1}(t)$ with a disjoint dense boundary subset of $(2k-3, 2k-2)$ constructed in Theorem 19; and if $t \in C_k^r$, then $f^{-1}(t)$ is the union of $f_k^{-1}(t) \cup f_{k+1}^{-1}(t)$ with a disjoint dense boundary subset of $(2k-1, 2k)$ constructed in the same theorem. In either case $f^{-1}(t) \in B(C^\infty(0, 2n-1))$. Therefore, $f^{-1}(t) \in B(C^\infty(0, 2n-1))$ for every $t \in \bigoplus_{k=1}^n Q_k$. Thus, the (**S4 + S5 + C**)-algebra over $\bigoplus_{k=1}^n Q_k$ is isomorphic to a subalgebra of $(B(C^\infty(0, 2n-1)), ^-, \exists)$, and so it is isomorphic to a subalgebra of $(B(C^\infty(\mathbb{R}), ^-, \exists)$. It follows that **S4 + S5 + C** is complete with respect to $(B(C^\infty(\mathbb{R})), ^-, \exists)$. □

5. Conclusions

In this paper we proved that **S4** is complete with respect to the closure algebra $(B(C^\infty(\mathbb{R})), ^{-})$. It follows that **S4** is complete with respect to any closure algebra containing $(B(C^\infty(\mathbb{R})), ^{-})$ and contained in $(\mathcal{P}(\mathbb{R}), ^{-})$. One closure algebra in the interval $[(B(C^\infty(\mathbb{R})), ^{-}), (\mathcal{P}(\mathbb{R}), ^{-})]$ deserves special mention. Let $\mathfrak{B}(\mathbb{R})$ denote the Boolean algebra of Borel sets over open subsets of $\mathbb{R}$; that is $\mathfrak{B}(\mathbb{R})$ is the countably complete Boolean algebra countably generated by $\mathcal{O}(\mathbb{R})$. It is obvious that $B(C^\infty(\mathbb{R})) \subseteq \mathfrak{B}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$. In fact, both of the inclusions are proper. As a result we obtain that **S4** is complete with respect to the closure algebra $(\mathfrak{B}(\mathbb{R}), ^{-})$.

In Remark 10 we pointed out that the modal system **Grz** is complete with respect to the closure algebra $(B(\mathcal{O}(\mathbb{R})), ^{-})$. It still remains an open problem to classify the complete logics of the closure algebras in between $(B(\mathcal{O}(\mathbb{R})), ^{-})$ and $(B(C^\infty(\mathbb{R})), ^{-})$.

In the language $\mathcal{L}(\forall)$ a natural extension of **Grz** is the bimodal system **Grz** + **S5** + **C**. However, it remains an open problem whether **Grz** + **S5** + **C** has the finite model property. Therefore, it is still an open problem whether **Grz** + **S5** + **C** is complete with respect to $(B(\mathcal{O}(\mathbb{R})), ^{-}, \exists)$.

Let $B(C(\mathbb{R}))$ denote the Boolean algebra generated by $C(\mathbb{R})$. It was proved in Aiello et al. [1] that the complete logic of $(B(C(\mathbb{R})), ^{-})$ is the complete logic of the closure algebra over the 2-tree of depth 2. This result was extended to the bimodal language $\mathcal{L}(\forall)$ in van Benthem et al. [15]. It still remains an open problem to classify the complete logics of the closure algebras in the interval $[(B(C(\mathbb{R})), ^{-}), (B(\mathcal{O}(\mathbb{R})), ^{-})]$, as well as the bimodal logics of the (**S4** + **S5** + **C**)-algebras in the interval $[(B(C(\mathbb{R})), ^{-}, \exists), (B(C^\infty(\mathbb{R})), ^{-}, \exists)]$.

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