Syntax of first order logic.

A first-order language $\mathcal{L}$ is a set $\{\dot{\mathrm{f}}_i\,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\,;\, j \in J\}$ of function symbols and relation symbols together with a signature $\sigma : I \cup J \to \mathbb{N}$. In addition to the symbols from $\mathcal{L}$, we shall be using the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\to$, $\neg$, $\leftrightarrow$, equality $=$, and a set of variables Var.

Definition of an $\mathcal{L}$-term.

- Every variable is an $\mathcal{L}$-term.
- If $\sigma(\dot{\mathrm{f}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{f}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-term.
- Nothing else is an $\mathcal{L}$-term.

Definition of an $\mathcal{L}$-formula.

- If t and t^* are $\mathcal{L}$-terms, then $t = t^*$ is an $\mathcal{L}$-formula.
- If $\sigma(\dot{\mathrm{R}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{R}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-formula.
- If φ and ψ are $\mathcal{L}$-formulae and x is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\forall x\,(\varphi)$ and $\exists x\,(\varphi)$ are $\mathcal{L}$-formulae.
- Nothing else is an $\mathcal{L}$-formula.

An $\mathcal{L}$-formula without free variables is called an $\mathcal{L}$-sentence.

Semantics of first order logic (1).

We fix a first-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i\,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\,;\, j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

A tuple $\mathbf{X} = \langle X, \langle f_i\,;\, i \in I\rangle, \langle R_j\,;\, j \in J\rangle\rangle$ is called an $\mathcal{L}$-structure if f_i is an $\sigma(\dot{\mathrm{f}}_i)$-ary function on X and R_i is an $\sigma(\dot{\mathrm{R}}_i)$-ary relation on X.

An X-interpretation is a function $\iota : \mathrm{Var} \to X$.

If ι is an X-interpretation and $\mathbf{X}$ is an $\mathcal{L}$ then ι extends to a function $\hat{\iota}$ on the set of all $\mathcal{L}$-terms.

If $\mathbf{X}$ is an $\mathcal{L}$-structure and ι is an X-interpretation, we defi ne a semantics for all $\mathcal{L}$-formulae by recursion.

Semantics of first order logic (2).

If $\mathbf{X}$ is an $\mathcal{L}$-structure and ι is an X-interpretation, we define a semantics for all $\mathcal{L}$-formulae by recursion.

- $\mathbf{X}, \iota \models t = t^*$ if and only if $\hat{\iota}(t) = \hat{\iota}(t^*)$.
- $\mathbf{X}, \iota \models \dot{\mathrm{R}}_j(t_1, ..., t_n)$ if and only if $R(\hat{\iota}(t_1), ..., \hat{\iota}(t_n))$.
- $\mathbf{X}, \iota \models \varphi \wedge \psi$ if and only if $\mathbf{X}, \iota \models \varphi$ and $\mathbf{X}, \iota \models \psi$.
- $\mathbf{X}, \iota \models \neg\varphi$ if and only if it is not the case that $\mathbf{X}, \iota \models \varphi$.
- $\mathbf{X}, \iota \models \forall x\,(\varphi)$ if and only if for all X-interpretations ι^* with $\iota \sim_x \iota^*$, we have $\mathbf{X}, \iota^* \models \varphi$.

- $\mathbf{X} \models \varphi$ if and only if for all X-interpretations ι, we have $\mathbf{X}, \iota \models \varphi$.

Object Language $\leftrightarrow$ Metalanguage.

Semantics of first order logic (3).

Object Language $\leftrightarrow$ Metalanguage.

Let $\mathbf{X}$ be an $\mathcal{L}$-structure. The theory of $\mathbf{X}$, $\mathrm{Th}(\mathbf{X})$, is the set of all $\mathcal{L}$-sentences φ such that $\mathbf{X} \models \varphi$.

Under the assumption that the *tertium non datur* holds for the metalanguage, the theory of $\mathbf{X}$ is always complete:

For every sentence φ, we either have $\varphi \in \mathrm{Th}(\mathbf{X})$ or $\neg\varphi \in \mathrm{Th}(\mathbf{X})$.

Deduction (1).

Let Φ be a set of $\mathcal{L}$-sentences. A Φ-proof is a finite sequence $\langle \varphi_1, ..., \varphi_n \rangle$ of $\mathcal{L}$-formulae such that for all i, one of the following holds:

- $\varphi_i \equiv t = t$ for some $\mathcal{L}$-term t,
- $\varphi_i \in \Phi$, or
- there are $j, k < i$ such that φ_j and φ_k are the premisses and φ_i is the conclusion in one of the rows of the following table.

Premisses		Conclusion
$\varphi \wedge \psi$		φ
$\varphi \wedge \psi$		ψ
φ	ψ	$\varphi \wedge \psi$
φ	$\neg\varphi$	ψ
$\varphi \to \psi$	$\neg\varphi \to \psi$	ψ
$\forall x\,(\varphi)$		$\varphi\frac{s}{x}$
$\varphi\frac{y}{x}$		$\forall x\,(\varphi))$
$t = t^*$	$\varphi\frac{t}{x}$	$\varphi\frac{t^*}{x}$

Deduction (2).

If Φ is a set of $\mathcal{L}$-sentences and φ is an $\mathcal{L}$-formula, we write $\Phi \vdash \varphi$ if there is a Φ-proof in which φ occurs.

We call a set Φ of sentences a theory if whenever $\Phi \vdash \varphi$, then $\varphi \in \Phi$ ("Φ is deductively closed").

Example. Let $\mathcal{L} = \{\leq\}$ be the language of partial orders. Let $\Phi_{\mathrm{p.o.}}$ be the axioms of partial orders, and let Φ be the deductive closure of $\Phi_{\mathrm{p.o.}}$. Φ is not a complete theory, as the sentence $\forall x \forall y (x \leq y \vee y \leq x)$ is not an element of Φ, but neither is its negation.

Completeness.

Kurt Gödel (1906-1978)

Semantic entailment. We write $\Phi \models \varphi$ for "whenever $\mathbf{X} \models \Phi$, then $\mathbf{X} \models \varphi$".

Gödel Completeness Theorem (1929).

$\Phi \vdash \varphi$ if and only if $\Phi \models \varphi$.

"there is a Φ-proof of φ" "for all $\mathbf{X} \models \Phi$, we have $\mathbf{X} \models \varphi$"

$\Phi \nvdash \varphi$ if and only if $\Phi \not\models \varphi$.

"no Φ-proof contains φ" "there is some $\mathbf{X} \models \Phi \wedge \neg\varphi$"

Applications (1).

The Model Existence Theorem.

If Φ is consistent (*i.e.*, $\Phi \nvdash \bot$), then there is a model $\mathbf{X} \models \Phi$.

The Compactness Theorem.

Let Φ be a set of sentences. If every fi nite subset of Φ has a model, then Φ has a model.

Proof. If Φ doesn't have a model, then it is inconsistent by the **Model Existence Theorem**. So, $\Phi \vdash \bot$, *i.e.*, there is a Φ-proof P of $\bot$.
But P is a finite object, so it contains only finitely many elements of Φ. Let Φ_0 be the set of elements occurring in P. Clearly, P is a Φ_0-proof of $\bot$, so Φ_0 is inconsistent. Therefore Φ_0 cannot have a model. q.e.d.

Applications (2).

The Compactness Theorem. Let Φ be a set of sentences. If every finite subset of Φ has a model, then Φ has a model.

Corollary 1. Let Φ be a set of sentences that has arbitrary large finite models. Then Φ has an infinite model.

Proof. Let $\psi_{\geq n}$ be the formula stating "there are at least n different objects". Let $\Psi := \{\psi_{\geq n} \,;\, n \in \mathbb{N}\}$. The premiss of the theorem says that every finite subset of $\Phi \cup \Psi$ has a model. By compactness, $\Phi \cup \Psi$ has a model. But this must be infinite. q.e.d.

Let $\mathcal{L} := \{\leq\}$ be the first order language with one binary relation symbol. Let $\Phi_{\mathrm{p.o.}}$ be the axioms of partial orders.

Corollary 2. There is no sentence σ such that for all partial orders $\mathbf{P}$, we have

$$\mathbf{P} \text{ is finite if and only if } \mathbf{P} \models \sigma.$$

[If σ is like this, then **Corollary 1** can be applied to $\Phi_{\mathrm{p.o.}} \cup \{\sigma\}$.]

Foundations of Mathematics.

- Does mathematics need foundations? (Not until 1900.)
- *Mathematical approach:* Work towards an axiom system of mathematics with purely mathematical means. (Hilbert's Programme). In its naïve interpretation crushed by Gödel's Incompleteness Theorem.
- *Extra-mathematical approach:* Use external arguments for axioms and rules: pragmatic, philosophical, sociological, (theological ?).
- Foundations of number theory: test case.

Sets are everything (1).

- Different areas of mathematics use different primitive notions: ordered pair, function, natural number, real number, transformation, *etc.*
- Set theory is able to incorporate all of these in one framework:
  - **Ordered Pair.** We defi ne

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}.$$

    (Kuratowski pair)
  - **Function.** A set f is called a function if there are sets X and Y such that $f \subseteq X \times Y$ and

$$\forall x, y, y' \left(\langle x, y \rangle \in f \& \langle x, y' \rangle \in f \to y = y' \right).$$

Sets are everything (2).

- Set theory incorporates basic notions of mathematics:
  - **Natural Numbers.** We call a set X inductive if it contains $\varnothing$ and for each $x \in X$, we have $x \cup \{x\} \in X$. Assume that there is an inductive set. Then defi ne $\mathbb{N}$ to be the intersection of all inductive sets.
  - **Rational Numbers.** We defi ne

$$\mathbb{P} := \{0,1\} \times \mathbb{N} \times \mathbb{N}\backslash\{0\}, \text{ then}$$

$$\langle i,n,m\rangle \sim \langle j,k,\ell\rangle :\iff i = j \;\&\; n \cdot \ell = m \cdot k, \text{ and}$$

$$\mathbb{Q} := \mathbb{P}/\sim.$$

Sets are everything (3).

- Set theory incorporates basic notions of mathematics:
  - **Real Numbers.** Define an order on $\mathbb{Q}$ by

$$\langle i, n, m \rangle \leq \langle j, k, \ell \rangle :\Longleftrightarrow\ i < j \vee (i = j \ \& \ n \cdot \ell \leq k \cdot m).$$

A subset X of $\mathbb{Q}$ is called an initial segment if

$$\forall x, y (x \in X \ \& \ y \leq x \rightarrow y \in X).$$

Initial segments are linearly ordered by inclusion. We define $\mathbb{R}$ to be the set of initial segments of $\mathbb{Q}$.

These definitions implicitly used a lot of set theoretic assumptions.

Sets.

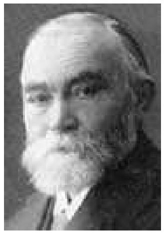
What is a set?

Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen. (Cantor 1895)

The Full Comprehension Scheme. Let X be our universe of discourse ("the universe of sets") and let Φ be any formula. Then the collection of those x such that $\Phi(x)$ holds is a set:

$$\{x\,;\;\Phi(x)\}.$$

Frege (1).



Gottlob Frege (1848-1925)

Frege's Comprehension Principle. If Φ is any formula, then there is some G such that

$$\forall x(G(x) \leftrightarrow \Phi(x)).$$

The ε operator. In Frege's system, we can assign to "concepts" F (second-order objects) a fi rst-order object εF ("the extension of F").

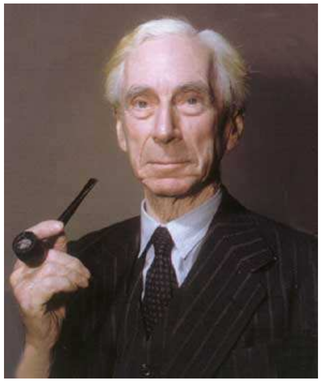
Frege (2).

Basic Law V. If F and G are concepts (second-order objects), then

$$\varepsilon F = \varepsilon G \quad \leftrightarrow \quad \forall x(F(x) \leftrightarrow G(x)).$$

Frege's Foundations of Arithmetic. Let F be an absurd concept ("round square"). Let G be the concept "being equinumerous to εF". We then defi ne $\mathbf{0} := \varepsilon G$. Suppose $\mathbf{0}$, ... , $\mathbf{n}$ are already defi ned. Then let H be the concept "being either $\mathbf{0}$ or ... or $\mathbf{n}$" and let $\overline{H}$ be the concept "being equinumerous to εH". Then let $\mathbf{n} + \mathbf{1} := \varepsilon \overline{H}$.

Russell (1).



Bertrand Arthur William
3rd Earl Russell (1872-1970)

- Grandson of John 1st Earl Russell (1792-1878); British prime minister (1846-1852 & 1865-1866).
- 1901: Russell discovers Russell's paradox.
- 1910-13: *Principia Mathematica* with Alfred North Whitehead (1861-1947).
- 1916: Dismissed from Trinity College for anti-war protests.
- 1918: Imprisoned for anti-war protests.
- 1940: Fired from City College New York.
- 1950: Nobel Prize for Literature.
- 1957: First Pugwash Conference.

Russell (2).

Frege's Comprehension Principle. Every formula defines a concept.
Basic Law V. If F and G are concepts, then $\varepsilon F = \varepsilon G \leftrightarrow \forall x(F(x) \leftrightarrow G(x))$.

Theorem (Russell). Basic Law V and the Full Comprehension Principle together are inconsistent.

Proof. Let R be the concept "being the extension of a concept which you don't fall under", *i.e.*, the concept described by the formula

$$\Phi(x) \ :\equiv \ \exists F(x = \varepsilon F \ \wedge \ \neg F(x)).$$

This concept exists by **Comprehension**. Let $r := \varepsilon R$.
Either $R(r)$ or $\neg R(r)$:

1. If $R(r)$, then there is some F such that $r = \varepsilon F$ and $\neg F(r)$. Thus $\varepsilon F = \varepsilon R$, and by **Basic Law V**, we have that $F(r) \leftrightarrow R(r)$. But then $\neg R(r)$. Contradiction!
2. If $\neg R(r)$, then for all F such that $r = \varepsilon F$ we have $F(r)$. But R is one of these F, so $R(r)$. Contradiction!

q.e.d.

Russell (3).

Theorem (Russell). The Full Comprehension Principle cannot be an axiom of set theory.

Proof. Suppose the Full Comprehension Principle holds, *i.e.*, every formula Φ describes a set $\{x\,;\,\Phi(x)\}$. Take the formula $\Phi(x) :\equiv x \notin x$ and form the set $r := \{x\,;\,x \notin x\}$ ("the Russell class").
Either $r \in r$ or $r \notin r$.

1. If $r \in r$, then $\Phi(r)$, so $r \notin r$. Contradiction!
2. If $r \notin r$, then $\neg\Phi(r)$, so $\neg r \notin r$, *i.e.*, $r \in r$. Contradiction!

q.e.d.

Frege & Russell.

- Russell discovered the paradox in June 1901.
- Russell's Paradox was discovered independently by Zermelo (Letter to Husserl, dated April 16, 1902).

  B. **Rang**, W. **Thomas**, Zermelo's discovery of the "Russell paradox", **Historia Mathematica** 8 (1981), p. 15-22.

- Letter to Frege (June 16, 1902) with the paradox.
- Frege's reply (June 22, 1902):

  > "with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish".

Attempts to resolve the paradoxes.

- **Theory of Types.**
  Russell (1903, “simple theory of types”; 1908, “ramifi ed theory of types”). *Principia Mathematica*.
- **Axiomatization of Set Theory.**
  Zermelo (1908). Skolem/Fraenkel (1922). Von Neumann (1925). “Zermelo-Fraenkel set theory” **ZF**.
- **Foundations of Mathematics.**
  Hilbert’s 2nd problem: *Consistency proof of arithmetic* (1900). Hilbert’s Programme (1920s).