UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2006/2007; 1st Semester
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Homework Set # 10 *Deadline:* November 29th, 2006

Exercise 33 (6 points).

We are modelling Achilles and the turtle as a transfinite process on the real line $\mathbb{R}$. Please give arguments for all answers.

(1) Achilles' position at time t is given by A_t, the turtle's position is given by T_t. We start with $A_0 := 0$ and $T_0 := 1$. For every index i, we define $A_{i+1} := A_i + |T_i - A_i|$, $T_{i+1} := T_i + \frac{1}{2} \cdot |T_i - A_i|$, and

$$T_\infty := \lim_{i \in \mathbb{N}} T_i,$$

$$A_\infty := \lim_{i \in \mathbb{N}} A_i,$$

$$T_{\infty+\infty} := \lim_{i \in \mathbb{N}} T_{\infty+i}, \text{ and}$$

$$A_{\infty+\infty} := \lim_{i \in \mathbb{N}} A_{\infty+i}.$$

Determine the least index i such that $A_i = T_i$ (1 point). Where is Achilles at time $\infty + \infty$ (1 point)?

(2) Now the positions are given by A^*_t and T^*_t defined as follows. For each index $i \in \{0, 1, 2, ..., \infty, \infty + 1, \infty + 2, \infty + 3, ...\}$, we define the *value* $\mathrm{v}(i)$ as follows:

$$\mathrm{v}(i) := n \text{ if } i = n \text{ or } i = \infty + n.$$

We start with $A^*_0 := 0$ and $T^*_0 := 1$. For every index i, we define $A^*_{i+1} := A^*_i + \frac{1}{2^{\mathrm{v}(i)}}$, $T^*_{i+1} := T^*_i + \frac{1}{2^{\mathrm{v}(i)+1}}$, and

$$T^*_\infty := \lim_{i \in \mathbb{N}} T^*_i,$$

$$A^*_\infty := \lim_{i \in \mathbb{N}} A^*_i,$$

$$T^*_{\infty+\infty} := \lim_{i \in \mathbb{N}} T^*_{\infty+i}, \text{ and}$$

$$A^*_{\infty+\infty} := \lim_{i \in \mathbb{N}} A^*_{\infty+i}.$$

Compute $A^*_{\infty+5}$, $T^*_{\infty+12}$, $A^*_{\infty+\infty}$ and $T^*_{\infty+\infty}$ (1 point each).

Exercise 34 (7 points).

Let $\mathcal{L} := \{+, \cdot, 0, 1, -\}$ be the language of Boolean algebras and Φ_{BA} be the axioms of Boolean algebras. Let

$$\varphi \;:\doteq\; \forall x \forall y \Big(\big((x \neq x \cdot y) \wedge (y \neq x \cdot y)\big) \rightarrow (x \cdot y = 0) \Big),$$

$$\psi \;:\doteq\; \exists x \big((x \neq 0) \wedge (x \neq 1)\big).$$

Let Φ_0, Φ_1, Φ_2, and Φ_3 be the deductive closures of Φ_{BA}, $\Phi_{\mathrm{BA}} \cup \{\neg\psi\}$, $\Phi_{\mathrm{BA}} \cup \{\varphi\}$, and $\Phi_{\mathrm{BA}} \cup \{\varphi, \psi\}$, respectively. Investigate whether Φ_i is a complete theory. If it isn't, give a formula σ such that $\sigma \notin \Phi_i$ and $\neg\sigma \notin \Phi_i$. If it is complete, give a brief argument why. (1 point each for Φ_0 and Φ_1, 2 points for Φ_2, 3 points for Φ_3.)

Exercise 35 (6 points).

(1) Give the names of the following logicians and mathematicians (1 point each):
- X was one of the students of David Hilbert who was a teacher at the *Gymnasium Arnoldinum* from 1929 to 1948.
- Y was an important figure in the history of the *Deutsche Mathematiker-Vereinigung*. He was married to the granddaughter of Hegel, and is popularly known for the "Y bottle", a two-dimensional manifold not embeddable into $\mathbb{R}^3$.

(2) Consider the following German mathematicians: Felix Bernstein, Ludwig Bieberbach, Kurt Schütte. Find out which of these is a student of David Hilbert (1 point per correct answer; please prove your answer by giving the year of the dissertation if the answer is "Yes" or the name of the PhD supervisor if the answer is "No").

(3) What is the canonical webpage for finding information about supervisor-student relations in mathematics? (1 point)

Exercise 36 (3 points).

(1) Find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \oplus \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \oplus \mathbf{W}$ and explain why (1½ points).

(2) Similarly, find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \otimes \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \otimes \mathbf{W}$ and explain why (1½ points).