UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
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Homework Set # 8 *Deadline:* April 13th, 2006

Exercise 23 (total of twelve points).

Reread the definition of $+$ on ordinals from Exercise 9 (HW Set # 3). We defined an addition operation (for reasons of notational clarity, let's call it $\boxplus$) by recursion as

$$n \boxplus 0 := n,$$
$$n \boxplus (m+1) := (n \boxplus m) + 1.$$

Prove that $\boxplus$ defines an associative (1½ points) and commutative (2½ points) operation. Prove that $+$ and $\boxplus$ are the same operation on ω (3 points).
Define a relation $\boxtimes$ on ordinals by letting $\alpha \boxtimes \beta$ be the least ordinal γ such that there is a bijection between γ and the set $\alpha \times \beta$. Compute $\omega \boxtimes n$ (1½ points) and $\omega \boxtimes \omega$ (1½ points). Prove that $\boxtimes$ is commutative (2 points).

Exercise 24 (total of twelve points).

If $\mathbf{X} := \langle X, \leq \rangle$ is a poset, we call $I \subseteq X$ an **initial segment** or **Dedekind cut** if for all $x \in I$ we have

$$\text{if } y \leq x, \text{ then } y \in I.$$

We call a Dedekind cut I **realized** if there is some $x \in X$ such that $I = X_x := \{y \in X\,;\, y < x\}$. We call $\mathbf{X}$ **Dedekind closed** if every Dedekind cut is realized.
Let $\mathrm{D}(\mathbf{X})$ be the poset of Dedekind cuts in $\mathbf{X}$ ordered by inclusion $\subseteq$.
Prove:

- For every toset $\mathbf{X}$, $\mathrm{D}(\mathbf{X})$ is Dedekind closed (3 points).
- Give an example of a poset $\mathbf{X}$ and a Dedekind cut I that is realized by two different elements, i.e., there are $x \neq x^*$ such that $I = X_x = X_{x^*}$ (3 points).
- Consider

  $$Y := \big(\{0\} \times [0,1)\big) \cup \big(\{1\} \times [0,1)\big) \cup \big(\{2\} \times (0,1]\big)$$

  ordered by $\langle i, x \rangle \leq \langle j, y \rangle \iff (i = j \wedge x \leq y) \vee (i = 0 \wedge j = 2) \vee (i = 1 \wedge j = 2)$. Draw a picture of Y (1 point). Show that $\mathbf{Y}$ is not Dedekind closed (1 point). If $X \supseteq Y$ and $\leq^* \cap Y \times Y = \leq$, we call $\mathbf{X} = \langle X, \leq^*$ an extension of $\mathbf{Y}$. Present two different extensions of $\mathbf{Y}$ that are both Dedekind complete but not isomorphic (2 points each).