UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatische Verzamelingentheorie

2005/2006; 2nd Semester
dr Benedikt Löwe

Homework Set # 2 *Deadline:* February 23rd, 2006

Exercise 5 (total of twelve points).

Let $\langle X, \leq \rangle$ be a wellordered set and Y be an arbitrary set. Show that the following are equivalent (7 points):

(1) There is a surjection from X to Y.
(2) There is an injection from Y to X.

Use the wellordering in order to give a concrete definition of the injection. Why could this be important? (1 point)
Let $f : Y \to X$ be an injection and define an ordering $\leq^*$ on Y by

$$y \leq^* y' :\iff f(y) \leq f(y').$$

Show that $\langle Y, \leq^* \rangle$ is a wellordered set. (4 points)

Exercise 6 (total of three points).

Let $\langle X, \leq \rangle$ and $\langle Y, \leq \rangle$ be two partially ordered sets. Remember that a function $f : X \to Y$ was called **orderpreserving** iff

$$x \leq x' \iff f(x) \leq f(x').$$

Let's define a weaker notion: we call f **monotone** if the implication

$$x \leq x' \Rightarrow f(x) \leq f(x').$$

Give an example of partial orders and a function that is monotone but not orderpreserving.

Exercise 7 (total of thirteen points).

In a total ordering $\langle X, \leq \rangle$, an **infinite ascending sequence** is a set $\{x_i \,;\, i \in \mathbb{N}\}$ such that for any i, we have $x_i < x_{i+1}$, an **infinite descending sequence** is a set $\{x_i \,;\, i \in \mathbb{N}\}$ such that for any i, we have $x_{i+1} < x_i$. If $A \subseteq X$ and $x \in X$, we say that x **lies above** A (x **lies below** A) if for all $a \in A$, we have $a < x$ ($x < a$).
Let $X := \mathbb{N}$. Define total orderings $\leq_0$, $\leq_1$, $\leq_2$ and $\leq_3$ on X such that

(1) X has an infinite descending sequence ($\leq_0$, 2 points),
(2) X has an infinite ascending sequence A, an element a that lies above A, and another infinite ascending sequence B such that a lies below B ($\leq_1$, 4 points),
(3) X has a least element, an infinite descending sequence A, an element a that lies above A, and both infinite ascending and descending sequences B and C such that a lies below both B and C ($\leq_2$, 4 points),
(4) X has a least and a biggest element and no infinite ascending sequences ($\leq_3$, 3 points).