The Continuum Hypothesis, Part II

W. Hugh Woodin

Introduction

In the first part of this article, I identified the *correct* axioms for the structure $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$, which is the standard structure for Second Order Number Theory. The axioms, collectively "Projective Determinacy", solve many of the otherwise unsolvable, classical problems of this structure.

Actually working from the axioms of set theory, ZFC, I identified a natural progression of structures increasing in complexity: $\langle H(\omega), \in \rangle$, $\langle H(\omega_1), \in \rangle$, and $\langle H(\omega_2), \in \rangle$, where for each cardinal κ, $H(\kappa)$ denotes the set of all sets whose transitive closure has cardinality less than κ. The first of these structures is logically equivalent to $\langle \mathbb{N}, +, \cdot \rangle$, the standard structure for number theory; the second is logically equivalent to the standard structure for Second Order Number Theory; and the third structure is where the answer to the Continuum Hypothesis, CH, lies. The main topic of Part I was the structure $\langle H(\omega_1), \in \rangle$.

Are there analogs of these axioms, say, some generalization of Projective Determinacy, for the structure $\langle H(\omega_2), \in \rangle$? Any reasonable generalization should settle the Continuum Hypothesis.

An immediate consequence of Cohen's method of forcing is that large cardinal axioms are not terribly useful in providing such a generalization. Indeed it was realized fairly soon after the discovery of forcing that essentially no large cardinal hypothesis can settle the Continuum Hypothesis. This was noted independently by Cohen and by Levy-Solovay.

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So the resolution of the theory of the structure $\langle H(\omega_2), \in \rangle$ could well be a far more difficult challenge than was the resolution of the theory of the structure $\langle H(\omega_1), \in \rangle$.

One example of the potential subtle aspects of the structure $\langle H(\omega_2), \in \rangle$ is given in the following theorem from 1991, the conclusion of which is in essence a property of the structure $\langle H(\omega_2), \in \rangle$.

Theorem (Woodin). *Suppose that the axiom Martin's Maximum holds. Then there exists a surjection $\rho : \mathbb{R} \to \omega_2$ such that $\{(x, y) \mid \rho(x) < \rho(y)\}$ is a projective set.* □

As we saw in Part I, assuming the forcing axiom, Martin's Maximum, CH holds projectively in that if $X \subseteq \mathbb{R}$ is an uncountable projective set, then $|X| = |\mathbb{R}|$. This is because Projective Determinacy must hold. However, the preceding theorem shows that assuming Martin's Maximum, CH fails projectively in that there exists a surjection

$$\rho : \mathbb{R} \to \omega_2$$

such that $\{(x, y) \mid \rho(x) < \rho(y)\}$ is a projective set. Such a function ρ is naturally viewed as a "projective counterexample" to CH, for it *is* a counterexample to the following reformulation of CH: Suppose that $\pi : \mathbb{R} \to \alpha$ is a surjection of $\mathbb{R}$ onto the ordinal α; then $\alpha < \omega_2$.

There is a curious asymmetry which follows from (the proofs of) these results. Assume there exist infinitely many Woodin cardinals. Then:

> **Claim (1)** There can be *no* projective "proof" of CH (there can be no projective well-ordering of $\mathbb{R}$ of length ω_1).

> **Claim (2)** There *can* be a projective "proof" of ¬CH (there can be, in the sense just defined, a projective counterexample to CH).

Therefore, if there exist infinitely many Woodin cardinals and if the Continuum Hypothesis is to be decided on the basis of "simple" evidence (i.e., projective evidence), then the Continuum Hypothesis must be *false*. This is the point of the first claim.

But is this an argument against CH? If so, the playful adversary might suggest that a similar line of argument indicates that ZFC is inconsistent, for while we can have a finite proof that ZFC is inconsistent, we, by Gödel's Second Incompleteness Theorem, cannot have a finite proof that ZFC is consistent (unless ZFC is inconsistent). There is a key difference here, though, which is the point of the second claim. If there is more than one Woodin cardinal, then a projective "proof" that CH is false can always be created by passing to a Cohen extension. More precisely, if $\langle M, E\rangle$ is a model of ZFC together with the statement "There exist 2 Woodin cardinals", that is, if

$$\langle M, E\rangle \vDash \text{ZFC+"There exist 2 Woodin cardinals,"}$$

then there is a Cohen extension, $\langle M^*, E^*\rangle$, of $\langle M, E\rangle$ such that $\langle M^*, E^*\rangle \vDash \text{ZFC} + \phi$, where ϕ is the sentence which asserts that there exists a surjection $\rho : \mathbb{R} \to \omega_2$ such that $\{(x, y) \mid \rho(x) < \rho(y)\}$ is a projective set. This theorem, which is the theorem behind the second claim above, shows that what might be called the *Effective Continuum Hypothesis* is as intractable as the Continuum Hypothesis itself.

These claims are weak evidence that CH is false, so perhaps large cardinal axioms are not quite so useless for resolving CH after all.

Of course there is no a priori reason that CH should be decided solely on the basis of projective evidence. Nevertheless, in the 1970s Martin conjectured that the existence of projective evidence against CH will eventually be seen to follow from *reasonable* axioms.

Axioms for $H(\omega_2)$

Encouraged by the success in Part I in finding the correct axioms for $H(\omega_1)$, and refusing to be discouraged by the observation that large cardinal axioms cannot settle CH, we turn our attention to $H(\omega_2)$. Here we have a problem if we regard large cardinal axioms as our sole source of inspiration: Even if there is an analog of Projective Determinacy for $H(\omega_2)$, how can we find it or even recognize it if we do find it?

My point is simply that the axiom(s) we seek cannot possibly be implied by any (consistent) large cardinal hypothesis remotely related to those currently accepted as large cardinal hypotheses.

Strong Logics

The solution is to take an abstract approach. This we shall do by considering strengthenings of first order logic and analyzing the following question, which I shall make precise.

> Can the theory of the structure $\langle H(\omega_2), \in\rangle$ be finitely axiomatized (over ZFC) in a (reasonable) logic which extends first order logic?

The logics arising naturally in this analysis satisfy two important conditions, *Generic Soundness* and *Generic Invariance*. As a consequence, any axioms we find will yield theories for $\langle H(\omega_2), \in\rangle$, whose "completeness" is immune to attack by applications of Cohen's method of forcing, just as is the case for number theory.

How shall we define the relevant *strong* logics? There is a natural strategy motivated by the Gödel Completeness Theorem. If ϕ is a sentence in the language $\mathcal{L}(\hat{=}, \hat{\in})$ for set theory, then "$\text{ZFC} \vdash \phi$" indicates that there is a formal proof of ϕ from ZFC. This is an arithmetic statement.

The Gödel Completeness Theorem shows that if ϕ is a sentence, then $\text{ZFC} \vdash \phi$ if and only if $\langle M, E\rangle \vDash \phi$ for *every* structure $\langle M, E\rangle$ such that $\langle M, E\rangle \vDash \text{ZFC}$.

Therefore a strong logic $\vdash_0$ can naturally be defined by first specifying a collection of *test* structures—these are structures of the form $\mathcal{M} = \langle M, E\rangle$, where $E \subset M \times M$—and then defining "$\text{ZFC} \vdash_0 \phi$" if for every test structure $\mathcal{M}$, if $\mathcal{M} \vDash \text{ZFC}$, then $\mathcal{M} \vDash \phi$.

Of course, we shall only be interested in the case that there actually exists a test structure $\mathcal{M}$ such that $\mathcal{M} \vDash \text{ZFC}$. In other words, we require that ZFC be *consistent* in our logic.

The *smaller* the collection of test structures, the *stronger* the logic, i.e., the larger the set of sentences ϕ which are proved by ZFC. Note that if there were only one test structure, then for each sentence ϕ either $\text{ZFC} \vdash_0 \phi$ or $\text{ZFC} \vdash_0 \neg\phi$. So in the logic $\vdash_0$ defined by this collection of test structures, *no* propositions are independent of the axioms ZFC.

By the Gödel Completeness Theorem, first order logic is the weakest (nontrivial) logic.

To formulate the notion of Generic Soundness, I first define the cumulative hierarchy of sets: this is a class of sets indexed by the ordinals. The set with index α is denoted V_α, and the definition is by induction on α as follows: $V_0 = \emptyset$; $V_{\alpha+1} = \mathcal{P}(V_\alpha)$; and if β is a limit ordinal, then $V_\beta = \cup\{V_\alpha \mid \alpha < \beta\}$. It is easily verified that the sets V_α are increasing, and it is a consequence of the axioms that every set is a member of V_α for large enough α.

It follows from the definitions that $V_\omega = H(\omega)$ and that $V_{\omega+1} \subseteq H(\omega_1)$. However, $V_{\omega+1} \neq H(\omega_1)$. Nevertheless, $V_{\omega+1}$ and $H(\omega_1)$ are *logically equivalent* in that each can be analyzed within the other. The relationship between $V_{\omega+2}$ and $H(\omega_2)$ is far more subtle. If the Continuum Hypothesis holds, then these structures are logically equivalent, but the assertion that these structures are logically equivalent does *not* imply the Continuum Hypothesis.

Suppose that M is a transitive set such that $\langle M, \in\rangle \vDash$ ZFC. The cumulative hierarchy in the sense of M is simply the sequence $M \cap V_\alpha$ indexed by $M \cap$ Ord. It is customary to denote $M \cap V_\alpha$ by M_α. If M is countable, then one can always reduce to considering Cohen extensions, M^*, which are transitive and for which the canonical embedding of M into M^* (given by Cohen's construction) is the identity. Thus, in this situation, $M \subseteq M^*$ and the ordinals of the Cohen extension coincide with those of the initial model.

The precise formulation of Generic Soundness involves notation from the *Boolean Valued Model* interpretation, due to Scott and Solovay, of Cohen's method of forcing. In the first part of this article I noted that Cohen extensions are parameterized by complete Boolean algebras (in the sense of the initial structure). Given a complete Boolean algebra $\mathbb{B}$, one can analyze *within* our universe of sets the Cohen extension of our universe that $\mathbb{B}$ could be used to define in some virtual larger universe where our universe, V, becomes, say, a countable transitive set. $V^{\mathbb{B}}$ denotes this potential extension, and for each ordinal α, $V^{\mathbb{B}}_\alpha$ denotes the α-th level of $V^{\mathbb{B}}$. For each sentence ϕ, the assertion "$V^{\mathbb{B}}_\alpha \vDash \phi$" is formally an assertion about the ordinal α and the Boolean algebra $\mathbb{B}$; this calculation is the essence of Cohen's method.

Definition. Suppose that $\vdash_0$ is a strong logic. The logic $\vdash_0$ satisfies *Generic Soundness* if for each sentence ϕ such that ZFC $\vdash_0 \phi$, the following holds. Suppose that $\mathbb{B}$ is a complete Boolean algebra, α is an ordinal, and $V^{\mathbb{B}}_\alpha \vDash$ ZFC. Then $V^{\mathbb{B}}_\alpha \vDash \phi$. □

Our context for considering strong logics will require at the very least that there exists a proper class of Woodin cardinals, and so the requirement of Generic Soundness is nontrivial. More precisely, assuming there exists a proper class of Woodin cardinals, for any complete Boolean algebra $\mathbb{B}$ there exist unboundedly many ordinals α such that $V^{\mathbb{B}}_\alpha \vDash$ ZFC.

The motivation for requiring Generic Soundness is simply that if ZFC $\vdash_0 \phi$, then the negation of ϕ should not be (provably) realizable by passing to a Cohen extension. Of course, if $\vdash_0$ is any strong logic which satisfies the condition of Generic Soundness, then it cannot be the case that either ZFC $\vdash_0$ CH or ZFC $\vdash_0 \neg$CH; i.e., CH remains unsolvable. This might suggest that an approach to resolving the theory of $H(\omega_2)$ based on strong logics is futile. But an important possibility arises through strong logics. This is the possibility that augmenting ZFC with a *single* axiom yields a system of axioms powerful enough to resolve, through inference in the strong logic, *all* questions about $H(\omega_2)$.

Definition. For a given strong logic $\vdash_0$, the theory of the structure $\langle H(\omega_2), \in\rangle$ is "*finitely axiomatized over* ZFC" if there exists a sentence Ψ such that for some α, $V_\alpha \vDash$ ZFC + Ψ, and for each sentence ϕ,

$$\text{ZFC} + \Psi \vdash_0 \text{ “}\langle H(\omega_2), \in\rangle \vDash \phi\text{”}$$

if and only if $\langle H(\omega_2), \in\rangle \vDash \phi$. □

Universally Baire Sets

There is a *transfinite* hierarchy which extends the hierarchy of the projective sets; this is the hierarchy of the *universally Baire* sets. Using these sets, I shall define a specific strong logic, Ω-logic.

Definition (Feng-Magidor-Woodin). A set $A \subseteq \mathbb{R}^n$ is *universally Baire* if for every continuous function

$$F : \Omega \to \mathbb{R}^n,$$

where Ω is a compact Hausdorff space, $F^{-1}[A]$ (the preimage of A by F) has the property of Baire in Ω; i.e., there exists an open set $O \subseteq \Omega$ such that the symmetric difference $F^{-1}[A] \vartriangle O$ is meager. □

It is easily verified that every Borel set $A \subseteq \mathbb{R}^n$ is universally Baire. More generally, the universally Baire sets form a σ-algebra closed under preimages by Borel functions

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

A little more subtle, and perhaps surprising, is that the universally Baire sets are Lebesgue measurable.

Every analytic set is universally Baire. The following theorem is proved using Jensen's *Covering Lemma.*

Theorem (Feng-Magidor-Woodin). *Suppose that every projective set is universally Baire. Then every analytic subset of* $[0, 1]$ *is determined.* □

The improvements of this theorem are quite subtle; the assumption that every projective set is universally Baire does *not* imply Projective Determinacy.

The following theorem of Neeman improves an earlier version of [Feng-Magidor-Woodin] which required the stronger hypothesis: There exist two Woodin cardinals.

Theorem (Neeman). *Suppose that there is a Woodin cardinal. Then every universally Baire subset of* $[0, 1]$ *is determined.* □

If there exists a proper class of Woodin cardinals, then the universally Baire sets are closed under continuous images (and so projections). Therefore:

Theorem. *Suppose that there are arbitrarily large Woodin cardinals. Then every projective set is universally Baire.* □

Sometimes the Euclidean space $\mathbb{R}$ is not the most illuminating space with which to deal. Let $\mathbb{K} \subseteq [0, 1]$

be the Cantor set, though any uncountable closed, nowhere dense subset of $[0,1]$ would suffice for what follows.

Suppose that $A \subseteq \mathbb{K}$ and that $B \subseteq \mathbb{K}$. The set A is *reducible* to B if there exists a continuous function

$$f : \mathbb{K} \to \mathbb{K}$$

such that $A = f^{-1}(B)$. The set A is *strongly reducible* to B if the function f can be chosen such that for all $x, y \in \mathbb{K}$, $|f(x) - f(y)| \le (1/2)|x - y|$.

There is a remarkably useful lemma of Wadge which I state for the projective subsets of the Cantor set and with the reducibilities just defined. There is a version of this lemma for subsets of $[0,1]$, but the definitions of reducible and strongly reducible must be changed.

Lemma (Wadge). *Suppose that the axiom Projective Determinacy holds and that A_0 and A_1 are projective subsets of $\mathbb{K}$. Then either A_0 is reducible to A_1 or A_1 is strongly reducible to $\mathbb{K}\backslash A_0$.* □

The proof of Wadge's lemma simply requires the determinacy of a set $B \subseteq [0,1]$ which is the preimage of

$$(A_0 \times (\mathbb{K}\backslash A_1)) \cup ((\mathbb{K}\backslash A_0) \times A_1)$$

by a Borel function $F : [0,1] \to \mathbb{R}^2$. Such sets B are necessarily universally Baire if both of the sets A_0 and A_1 are universally Baire. Therefore, by Neeman's theorem:

Theorem. *Suppose that there is a Woodin cardinal. Suppose that A_0 and A_1 are universally Baire subsets of $\mathbb{K}$. Then either A_0 is reducible to A_1 or A_1 is strongly reducible to $\mathbb{K}\backslash A_0$.* □

Suppose that $f : \mathbb{K} \to \mathbb{K}$ is such that for all $x, y \in \mathbb{K}$, $|f(x) - f(y)| \le (1/2)|x - y|$. Then for some $x_0 \in \mathbb{K}$, $f(x_0) = x_0$. This implies that no set $A \subseteq \mathbb{K}$ can be strongly reducible to its complement $\mathbb{K}\backslash A$. Therefore, given two universally Baire subsets of $\mathbb{K}$, A_0 and A_1, and assuming there is a Woodin cardinal, exactly one of the following must hold. This is easily verified by applying the previous theorem to the relevant pairs of sets, sorting through the various possibilities, and eliminating those that lead to the situation that a set is strongly reducible to its complement.

1. Both A_0 and $\mathbb{K}\backslash A_0$ are strongly reducible to A_1, and A_1 is not reducible to A_0 (or to $\mathbb{K}\backslash A_0$).
2. Both A_1 and $\mathbb{K}\backslash A_1$ are strongly reducible to A_0, and A_0 is not reducible to A_1 (or to $\mathbb{K}\backslash A_1$).
3. A_0 and A_1 are reducible to each other, or $\mathbb{K}\backslash A_0$ and A_1 are reducible to each other.

Thus one can define an equivalence relation on the universally Baire subsets of the Cantor set by $A_0 \sim_{\mathrm{w}} A_1$ if (3) holds, and one can totally order the induced equivalence classes by defining for universally Baire sets, A_0 and A_1, $A_0 <_{\mathrm{w}} A_1$ if (1) holds.

Of course (1) can be used to define a partial order on all subsets of $\mathbb{K}$. In the context of determinacy assumptions, Martin proved that this partial order is well founded [Moschovakis 1980]. In the absence of any determinacy assumptions, Martin's theorem can be formulated as follows.

Theorem (Martin). *Suppose that $\langle A_k : k \in \mathbb{N}\rangle$ is a sequence of subsets of $\mathbb{K}$ such that for all $k \in \mathbb{N}$, both A_{k+1} and $\mathbb{K}\backslash A_{k+1}$ are strongly reducible to A_k. Then there exists a continuous function $g : \mathbb{K} \to \mathbb{K}$ such that $g^{-1}(A_1)$ does not have the property of Baire.* □

As a corollary we obtain the well-foundedness of $<_{\mathrm{w}}$, because the continuous preimages of a universally Baire set must have the property of Baire.

So (assuming large cardinals) the universally Baire subsets of the Cantor set form a well-ordered hierarchy under a suitable notion of complexity. The projective sets define an initial segment, since any set which is reducible to a projective set is necessarily a projective set. The hierarchy finely calibrates the universally Baire sets. For example, the initial segment of length ω_1 is given by the Borel sets, and the corresponding ordinal rank of a Borel set is closely related to its classical Borel rank.

There is a natural generalization of first order logic which is defined from the universally Baire sets. This is Ω-logic; the "proofs" in Ω-logic are witnessed by universally Baire sets which can be assumed to be subsets of the Cantor set $\mathbb{K}$. The ordinal rank of the witness in the hierarchy of such sets I have just defined provides a quite reasonable notion of the length of a proof in Ω-logic.

The definition of Ω-logic involves the notion of an A-closed transitive set where A is universally Baire.

A-closed Sets

Suppose that M is a transitive set with the property that $\langle M, \in\rangle \vDash \text{ZFC}$.

Suppose that $(\Omega, F, \tau) \in M$ and that

1. $\langle M, \in\rangle \vDash$ "Ω is a compact Hausdorff space".
2. τ is the topology on Ω; i.e., τ is the set of $O \in M$ such that $\langle M \in\rangle \vDash$ "$O \subseteq \Omega$ and O is open".
3. $\langle M, \in\rangle \vDash$ "$F \in C(\Omega, \mathbb{R})$".

For example, if M is countable and

$$\langle M, \in\rangle \vDash \text{"}\Omega \text{ is the unit interval } [0,1]\text{"},$$

then $\Omega = [0,1] \cap M$. It is easily verified that $[0,1] \cap M$ is dense in $[0,1]$, and so in this case Ω is a countable dense subspace of $[0,1]$.

Notice that the element F of M *is* necessarily a function, $F : \Omega \to \mathbb{R}$.

Trivially, τ is a base for a topology on Ω yielding a topological space which of course need not be compact, as the preceding example illustrates. Nevertheless, this topological space is necessarily completely regular. The function F is easily seen to be continuous on this space.

Let $\tilde{\Omega}$ be the Stone-Čech compactification of this space, and for each set $O \in \tau$ let $\tilde{O}$ be the open subset of $\tilde{\Omega}$ defined by O. This is the complement of the closure, computed in $\tilde{\Omega}$, of $\Omega \backslash O$.

The function F has a unique continuous extension

$$\tilde{F} : \tilde{\Omega} \to \mathbb{R}.$$

Suppose $A \subseteq \mathbb{R}$ is universally Baire. Then the preimage of A under $\tilde{F}$ has the property of Baire in $\tilde{\Omega}$. Let

$$\tau_A = \{O \in \tau \mid \tilde{O} \backslash \tilde{F}^{-1}[A] \text{ is meager}\}.$$

Definition. Suppose that $A \subseteq \mathbb{R}$ is universally Baire and that M is a transitive set with $\langle M, \in\rangle \vDash \text{ZFC}$. The set M is *A-closed* if for every $(\Omega, F, \tau) \in M$ as above, $\tau_A \in M$. □

If M is A-closed, then $A \cap M \in M$, but in general the converse fails.

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then there exists a universally Baire set $A^* \subseteq \mathbb{K}$ such that for all transitive sets M such that $\langle M, \in\rangle \vDash \text{ZFC}$, M is A-closed if and only if M is A^*-closed. Thus, for our purposes, the distinction between universally Baire subsets of $\mathbb{R}$ versus universally Baire subsets of $\mathbb{K}$, the Cantor set, is not relevant.

Ω-logic

Having defined A-closure, I can now define Ω-logic. This logic can be defined without the large cardinal assumptions used here, but the definition becomes a bit more technical.

Definition. Suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence. Then

$$\text{ZFC} \vdash_\Omega \phi$$

if there exists a universally Baire set $A \subseteq \mathbb{R}$ such that $\langle M, \in\rangle \vDash \phi$ for every countable transitive A-closed set M such that $\langle M, \in\rangle \vDash \text{ZFC}$. □

There are only countably many sentences in the language $\mathcal{L}(\hat{=}, \hat{\in})$, and, further, the universally Baire sets are closed under countable unions and preimages by Borel functions. Therefore there must exist a single universally Baire set $A_\Omega \subseteq \mathbb{R}$ such that for all sentences ϕ of $\mathcal{L}(\hat{=}, \hat{\in})$, $\text{ZFC} \vdash_\Omega \phi$ if and only if $\langle M, \in\rangle \vDash \phi$ for every countable transitive set M such that M is A_Ω-closed and $\langle M, \in\rangle \vDash \text{ZFC}$. Thus Ω-logic is the strong logic defined by taking as the collection of test structures the countable transitive sets M such that $\langle M, \in\rangle$ is a model of ZFC and M is A_Ω-closed.

One can easily generalize the definition of Ω-logic to define when $T \vdash_\Omega \phi$ where T is an arbitrary theory containing ZFC. If T is simply ZFC + Ψ for some sentence Ψ, then $T \vdash_\Omega \phi$ if and only if $\text{ZFC} \vdash_\Omega (\Psi \to \phi)$.

Suppose that ϕ is a sentence (of $\mathcal{L}(\hat{=}, \hat{\in})$) and that $\text{ZFC} \vdash_\Omega \phi$. Suppose that $A \subseteq \mathbb{K}$ is a universally Baire set which witnesses this. Viewing A as a "proof", one can naturally define the "length" of this proof to be the ordinal of A in the hierarchy of the universally Baire subsets of the Cantor set, $\mathbb{K}$, given by the relation $<_{\text{w}}$.

Thus one can define the usual sorts of Gödel and Rosser sentences. These are "self-referential sentences", and Rosser's construction yields sentences with stronger undecidability properties than does Gödel's construction.

For example, one can construct a sentence ϕ_0 (obviously false) which expresses:

> "There is a proof $\text{ZFC}^+ \vdash_\Omega (\neg\phi_0)$ for which there is no shorter proof $\text{ZFC}^+ \vdash_\Omega \phi_0$",

where ZFC^+ is ZFC together with the axiom "There exists a proper class of Woodin cardinals". Such constructions illustrate that Ω-logic is a reasonable generalization of first order logic.

Later in this article I shall make use of the notion of the length of a proof in Ω-logic when I define abstractly the hierarchy of large cardinal axioms.

Ω-logic is unaffected by passing to a Cohen extension. This is the property of *Generic Invariance*. The formal statement of this theorem involves some notation, which I discuss. It is customary in set theory to write for a given sentence ϕ, "$V \vDash \phi$" to indicate that ϕ is true, i.e., true in V, the universe of sets. Similarly, if $\mathbb{B}$ is a complete Boolean algebra, "$V^{\mathbb{B}} \vDash \phi$" indicates that ϕ is true in the Cohen extensions of V that $\mathbb{B}$ could be used to define (again in some virtual universe where our universe becomes a countable transitive set, as briefly discussed when the notation "$V_\alpha^{\mathbb{B}} \vDash \phi$" was introduced just before the definition of Generic Soundness).

Theorem (Generic Invariance). *Suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence. Then for each complete Boolean algebra $\mathbb{B}$, $\text{ZFC} \vdash_\Omega \phi$ if and only if $V^{\mathbb{B}} \vDash$"$\text{ZFC} \vdash_\Omega \phi$".* □

Similar arguments establish that if there exists a proper class of Woodin cardinals, then Ω-logic satisfies Generic Soundness.

The following theorem is a corollary of results mentioned in the first part of this article.

Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for each sentence ϕ,*

$$\text{ZFC} \vdash_\Omega \text{"}\langle H(\omega_1), \in\rangle \vDash \phi\text{"}$$

if and only if $\langle H(\omega_1), \in\rangle \vDash \phi$. □

A straightforward corollary is that

$$\text{ZFC} \vdash_\Omega \text{ Projective Determinacy},$$

which vividly illustrates that Ω-logic is stronger than first order logic.

The question of whether there can exist analogs of determinacy for the structure $\langle H(\omega_2), \in\rangle$ can now be given a precise formulation.

> Can there exist a sentence Ψ such that for all sentences ϕ either
>
> $$\text{ZFC} + \Psi \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \models \phi\text{” or}$$
>
> $$\text{ZFC} + \Psi \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \models \neg\phi\text{”}$$
>
> and such that ZFC + Ψ is Ω-consistent?

Such sentences Ψ will be candidates for the generalization of Projective Determinacy to $H(\omega_2)$. Notice that I am *not* requiring that the sentence Ψ be a proposition about $H(\omega_2)$; the sentence can refer to arbitrary sets.

Why seek such sentences?

Here is why. By adopting axioms which "settle" the theory of $\langle H(\omega_2), \in\rangle$ in Ω-logic, one recovers for the theory of this structure the *empirical* completeness currently enjoyed by number theory. This is because of the generic invariance of Ω-logic.

More speculatively, such axioms might allow for the development of a truly rich theory for the structure $\langle H(\omega_2), \in\rangle$, free to a large extent from the ubiquitous occurrence of unsolvable problems. Compare, for example, the theory of the projective sets as developed under the assumption of Projective Determinacy with the theory developed of the problems about the projective sets which are not solvable simply from ZFC.

The ideal $\mathcal{I}_{\text{NS}}$, which I now define, plays an essential and fundamental role in the usual formulation of Martin's Maximum.

Definition. $\mathcal{I}_{\text{NS}}$ is the σ-ideal of all sets $A \subseteq \omega_1$ such that $\omega_1 \backslash A$ contains a closed unbounded set. A set $S \subseteq \omega_1$ is *stationary* if for each closed, unbounded set $C \subseteq \omega_1$, $S \cap C \neq \emptyset$. A set $S \subseteq \omega_1$ is *co-stationary* if the complement of S is stationary. □

The countable additivity and the nonmaximality of the ideal $\mathcal{I}_{\text{NS}}$ are consequences of the Axiom of Choice.

In my view, the continuum problem is a direct consequence of assuming the Axiom of Choice. This is simply because by assuming the Axiom of Choice, the reals can be well ordered and so $|\mathbb{R}| = \aleph_\alpha$ for some ordinal α. Which α? This is the continuum problem.

Arguably, the stationary, co-stationary, subsets of ω_1 constitute the simplest true manifestation of the Axiom of Choice. A metamathematical analysis shows that assuming Projective Determinacy, there is really no manifestation within $H(\omega_1)$ of the Axiom of Choice. More precisely, the analysis of the projective sets, assuming Projective Determinacy, does not require the Axiom of Choice.

These considerations support the claim that the structure $\langle H(\omega_2), \in\rangle$ is indeed the *next* structure to consider after $\langle H(\omega_1), \in\rangle$, being the simplest structure where the influence of the Axiom of Choice is manifest.

The Axiom (∗)

I now come to a central definition, which is that of the axiom (∗). This axiom is a candidate for the generalization of Projective Determinacy to the structure $\langle H(\omega_2), \in\rangle$. The definition of the axiom (∗) involves some more notation from the syntax of formal logic. It is frequently important to monitor the complexity of a formal sentence. This is accomplished through the Levy hierarchy.

The collection of Σ_0 formulas of our language $\mathcal{L}(\hat{=}, \hat{\in})$ is defined as the smallest set of formulas which contains all quantifier-free formulas and which is closed under the application of *bounded quantifiers*.

Thus, if ψ is a Σ_0 formula, then so are the formulas $(\forall x_i((x_i \hat{\in} x_j) \to \psi))$ and $(\exists x_i((x_i \hat{\in} x_j) \wedge \psi))$.

We shall be interested in formulas which are of the form $(\forall x_i(\exists x_j \psi))$, where ψ is a Σ_0 formula. These are the Π_2 formulas. Somewhat simpler are the Π_1 formulas and the Σ_1 formulas; these are the formulas of the form $(\forall x_i \psi)$ or of the form $(\exists x_i \psi)$ respectively, where ψ is again a Σ_0 formula.

Informally, a Π_2 sentence requires two (nested) "unbounded searches" to verify that the sentence is true, whereas for a Π_1 sentence only one unbounded search is required. Verifying that a Σ_1 sentence is true is even easier.

For example, consider the structure $\langle H(\omega), \in\rangle$, which I have already noted is in essence the standard structure for number theory.

Many of the famous conjectures of modern mathematics are expressible as Π_1 sentences in this structure. This includes both Goldbach's Conjecture and the Riemann Hypothesis. However, the Twin Prime Conjecture is expressible by a Π_2 sentence, as is, for example, the assertion that $\text{P} \neq \text{NP}$, and neither is obviously expressible by a Π_1 sentence. This becomes interesting if, say, either of these latter problems were proved to be unsolvable from, for example, the natural axioms for $\langle H(\omega), \in\rangle$. Unlike the unsolvability of a Π_1 sentence, from which one can infer its "truth", for Π_2 sentences the unsolvability does not immediately yield a resolution.

If M is a transitive set and P and Q are subsets of M, then one may consider $\langle M, P, Q, \in\rangle$ as a structure for the language $\mathcal{L}(\hat{=}, \hat{P}, \hat{Q}, \hat{\in})$, obtained by adding two new symbols, $\hat{P}$ and $\hat{Q}$, to $\mathcal{L}(\hat{=}, \hat{\in})$. One defines the Σ_0 formulas and the Π_2 formulas of this expanded language in the same way as above.

The structure I actually wish to consider is

$$\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle,$$

where $X \subseteq \mathbb{R}$ is universally Baire. If ϕ is a sentence in the language $\mathcal{L}(\hat{=}, \hat{P}, \hat{Q}, \hat{\in})$ for this structure, then there is a natural interpretation of the assertion that

$$\text{ZFC} + \text{“}\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle \models \phi\text{”}$$

is Ω-consistent. The only minor problem is how to deal with X. But X is universally Baire. Thus I define

$$\text{ZFC} + \text{“}\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle \models \phi\text{”}$$

to be Ω-consistent if for every universally Baire set A there exists a countable transitive set M such that

1. M is A-closed and M is X-closed;
2. $\langle M, \in\rangle \models \text{ZFC}$;
3. $\langle H(\omega_2)^M, (\mathcal{I}_{\text{NS}})^M, X \cap M, \in\rangle \models \phi$, where

$$H(\omega_2)^M = \{a \in M \mid \langle M, \in\rangle \models \text{“}a \in H(\omega_2)\text{”}\},$$

and

$$(\mathcal{I}_{\text{NS}})^M = \{a \in M \mid \langle M, \in\rangle \models \text{“}a \in \mathcal{I}_{\text{NS}}\text{”}\}.$$

These are the relevant sets as computed in M.

With these definitions in hand, I come to the definition of the axiom $(*)$. The version I give is anchored in the projective sets; stronger versions of the axiom are naturally obtained by allowing more universally Baire sets in the definition.

> ***Axiom (*):*** There is a proper class of Woodin cardinals, and for each projective set $X \subseteq \mathbb{R}$, for each Π_2 sentence ϕ, if the theory
>
> $$\text{ZFC+ “}\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle \models \phi\text{”}$$
>
> is Ω-consistent, then
>
> $$\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle \models \phi.$$

What kinds of assertions are there which can be formulated in the form $\langle H(\omega_2), \in\rangle \models \phi$ for some Π_2 sentence ϕ? There are many examples. One example is Martin's Axiom (ω_1). Another, which identifies a consequence of the axiom $(*)$, is the assertion that if $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ are each nowhere countable and of cardinality $\aleph_1$, then A and B are order isomorphic. A set $X \subset \mathbb{R}$ is nowhere countable if $X \cap O$ is uncountable for each (nonempty) open set $O \subseteq \mathbb{R}$. Thus, assuming the axiom $(*)$, there is exactly *one* possible order type for nowhere countable subsets of $\mathbb{R}$ which have cardinality $\aleph_1$. I refer the reader to [Shelah 1998] for details, references, and other examples.

The axiom $(*)$ is really a maximality principle somewhat analogous to asserting algebraic closure for a field.

A cardinal κ is an *inaccessible cardinal* if it is a limit cardinal with the additional property that any cofinal subset of κ necessarily has cardinality κ. For example, ω is an inaccessible cardinal. The axiom which asserts the existence of an uncountable inaccessible cardinal is the weakest traditional large cardinal axiom.

Theorem. *Suppose that there exists a proper class of Woodin cardinals and that there is an inaccessible cardinal which is a limit of Woodin cardinals. Then* ZFC + axiom $(*)$ *is Ω-consistent.* □

There is an elaborate machinery of iterated forcing: this is the technique of iterating Cohen's method of building extensions [Shelah 1998]. It is through application of this machinery that, for example, the consistency of

$$\text{ZFC} + \text{“Martin's Maximum”}$$

is established (assuming the consistency of ZFC together with a specific large cardinal axiom, much stronger than, for example, the axiom that there is a Woodin cardinal).

Iterated forcing can be used to show the consistency of statements of the form "$\langle H(\omega_2), \in\rangle \models \phi$" for a rich variety of Π_2 sentences ϕ.

The previous theorem, on the Ω-consistency of the axiom $(*)$, is proved using the method of forcing but *not* using any machinery of iterated forcing. Further, the theorem is not proved as a corollary of some deep analysis of which Π_2 sentences can hold in $H(\omega_2)$.

The axiom $(*)$ settles in Ω-logic the *full* theory of the structure $\langle H(\omega_2), \in\rangle$. The stronger version of the following theorem, obtained by replacing $\langle H(\omega_2), \in\rangle$ with $\langle H(\omega_2), X, \in\rangle$ where X is a projective set, is also true.

Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for each sentence ϕ, either*

$$\text{ZFC} + \text{axiom } (*) \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \models \phi\text{” } \textit{or}$$

$$\text{ZFC} + \text{axiom } (*) \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \models \neg\phi\text{”}. \quad \square$$

Suppose that ϕ is a Π_2 sentence and that $X \subseteq \mathbb{R}$ is a projective set such that

$$\text{ZFC + “}\langle H(\omega_2), \mathcal{I}_{\text{NS}}, X, \in\rangle \models \phi\text{”}$$

is *not* Ω-consistent. Then the analysis behind the proof of the Ω-consistency of the axiom $(*)$ yields a projective witness for the corresponding Ω-proof.

Thus the axiom $(*)$ is in essence an axiom which can be localized to $H(\omega_2)$. More precisely, there is a (recursive) set of axioms, i.e., a recursive theory T, such that, assuming the existence of a proper class of Woodin cardinals, the axiom $(*)$ holds if and only if

$$\langle H(\omega_2), \in\rangle \models T.$$

Finally, assuming there is a proper class of Woodin cardinals, the axiom $(*)$ is equivalent to a strong form of a bounded version of Martin's Maximum, so again seemingly disparate threads are woven into a single tapestry.

The Axiom $(*)$ and $2^{\aleph_0}$

There is a Π_2 sentence ψ_{AC}, which if true in the structure $\langle H(\omega_2), \in\rangle$ implies that $2^{\aleph_0} = \aleph_2$.

The statement "$\langle H(\omega_2), \in\rangle \vDash \psi_{\mathrm{AC}}$" is Ω-consistent, and so as a corollary the axiom $(*)$ *implies* $2^{\aleph_0} = \aleph_2$.

Definition ψ_{AC}: Suppose S and T are each stationary, co-stationary, subsets of ω_1. Then there exist: a closed unbounded set $C \subseteq \omega_1$; a well-ordering $\langle L, <\rangle$ of cardinality ω_1; and a bijection $\pi : \omega_1 \to L$ such that for all $\alpha \in C$,

$$\alpha \in S \leftrightarrow \alpha^* \in T,$$

where α^* is the countable ordinal given by the order type of $\{\pi(\beta) \mid \beta < \alpha\}$ as a suborder of $\langle L, <\rangle$. □

By standard methods ψ_{AC} can be shown to be expressible in the required form (as a Π_2 sentence).

Lemma. *Suppose that ψ_{AC} holds. Then $2^{\aleph_0} = \aleph_2$.* □

There is a subtle aspect to this lemma. Suppose that CH holds and that $\langle x_\alpha : \alpha < \omega_1\rangle$ is an enumeration of $\mathbb{R}$.

Thus ψ_{AC} must fail. However, it is possible that there is *no* counterexample to ψ_{AC} which is *definable* from the given enumeration $\langle x_\alpha : \alpha < \omega_1\rangle$.

I note that Martin's Maximum can be shown to imply ψ_{AC}. This gives a completely different view of why the axiom Martin's Maximum implies that $2^{\aleph_0} = \aleph_2$.

And What about CH?

The basic question is the following. Is there an analog of the axiom $(*)$ in the context of CH? Continuing the analogy with the theory of fields, one seeks to complete the similarity:

$$\frac{? + \mathrm{CH}}{\text{axiom } (*)} \sim \frac{\text{real closed } + \text{ ordered}}{\text{algebraically closed}}.$$

More generally: Under what circumstances can the theory of the structure $\langle H(\omega_2), \in\rangle$ be finitely axiomatized, over ZFC, in Ω-logic?

Formally the sentences of our language, $\mathcal{L}(\hat{=}, \hat{\in})$, are (certain) finite sequences of elements of the underlying alphabet, which in this case can be taken to be $\mathbb{N}$. There is a natural (recursive) bijection of $\mathbb{N}$ with the set of all finite sequences from $\mathbb{N}$. This associates to each sentence ϕ of $\mathcal{L}(\hat{=}, \hat{\in})$ a positive integer k_ϕ, which is the *Gödel number of ϕ*.

To address the questions above, I require a definition generalizing the definition of $0'$ where $0'$ is the set

$$\{k_\phi \mid \phi \text{ is a } \Sigma_1 \text{ sentence and } \langle H(\omega), \in\rangle \vDash \phi\}.$$

Assuming ZFC is consistent, then the set

$$\{k_\phi \mid \phi \text{ is a sentence and ZFC} \vdash \phi\}$$

is *recursively* equivalent to $0'$ (a simple, though somewhat subtle, claim). In fact, one could reasonably take this as the definition of $0'$. This suggests the definition of $0^{(\Omega)}$.

Definition. Suppose that there exists a proper class of Woodin cardinals. Then

$$0^{(\Omega)} = \{k_\phi \mid \mathrm{ZFC} \vdash_\Omega \phi\}. \qquad \square$$

Suppose that M is a transitive set and $A \subseteq M$. Then the set A is *definable* in the structure $\langle M, \in\rangle$ if there is a formula $\psi(x_1)$ of $\mathcal{L}(\hat{=}, \hat{\in})$ such that

$$A = \{a \mid \langle M, \in\rangle \vDash \psi[a]\}.$$

The following theorem is one version of Tarski's theorem on the undefinability of truth.

Theorem (Tarski). *Suppose that M is a transitive set with $H(\omega) \subseteq M$. Then the set $\{k_\phi \mid \langle M, \in\rangle \vDash \phi\}$ is not definable in the structure $\langle M, \in\rangle$.* □

For each sentence Ψ, the set $\{k_\phi \mid \mathrm{ZFC} + \Psi \vdash \phi\}$ is definable in the structure $\langle H(\omega), \in\rangle$. Thus by Tarski's theorem, for each sentence Ψ the set

$$\{\phi \mid \mathrm{ZFC} + \Psi \vdash \text{“}\langle H(\omega), \in\rangle \vDash \phi\text{”}\}$$

is *not* equal to the set $\{\phi \mid \langle H(\omega), \in\rangle \vDash \phi\}$. This is a special case of Gödel's First Incompleteness Theorem.

Analogous considerations apply in our situation, and so our basic problem of determining when the theory of the structure $\langle H(\omega_2), \in\rangle$ can be finitely axiomatized, over ZFC, in Ω-logic naturally leads to the problem: How complicated is $0^{(\Omega)}$?

This set looks potentially extremely complicated, for it is in essence Ω-logic. Note that since

$$\{\phi \mid \mathrm{ZFC} \vdash_\Omega \text{“}\langle H(\omega_1), \in\rangle \vDash \phi\text{”}\}$$

is equal to the set $\{\phi \mid \langle H(\omega_1), \in\rangle \vDash \phi\}$ by Tarski's theorem, $0^{(\Omega)}$ is *not* definable in the structure $\langle H(\omega_1), \in\rangle$.

The calculation of the complexity of $0^{(\Omega)}$ involves adapting the Inner Model Program to analyze models of *Determinacy Axioms* rather than models of *Large Cardinal Axioms*.

This analysis, which is a bit involved and technical, yields the following result where c^+ denotes the least cardinal greater than c. Suppose that there exists a proper class of Woodin cardinals. Then $0^{(\Omega)}$ is definable in the structure $\langle H(c^+), \in\rangle$. Now if the Continuum Hypothesis holds, then $c = \omega_1$ and so $H(c^+) = H(\omega_2)$. Therefore, if the Continuum Hypothesis holds, then $0^{(\Omega)}$ is definable in the structure $\langle H(\omega_2), \in\rangle$.

Appealing to Tarski's theorem, we obtain as a corollary our main theorem.

Theorem. *Suppose that there exists a proper class of Woodin cardinals, $V_\kappa \vDash \mathrm{ZFC} + \Psi$, and for each sentence ϕ of $\mathcal{L}(\hat{=}, \hat{\in})$ either*

$$\mathrm{ZFC} + \Psi \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \vDash \phi\text{” } or$$

$$\mathrm{ZFC} + \Psi \vdash_\Omega \text{“}\langle H(\omega_2), \in\rangle \vDash \neg\phi\text{”}.$$

Then CH *is false.* □

There are more precise calculations of the complexity of $0^{(\Omega)}$ than I have given. For the indicated application on CH, one is actually interested in the complexity of sets $X \subseteq \mathbb{N}$ which are *Ω-recursive*. Ultimately, it is not really CH which is the critical issue, but effective versions of CH.

The Ω Conjecture

Perhaps Ω-logic is not the *strongest* reasonable logic.

Definition (Ω*-logic). Suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence. Then

$$\text{ZFC} \vdash_{\Omega^*} \phi$$

if for all ordinals α and for all complete Boolean algebras $\mathbb{B}$, if $V_\alpha^{\mathbb{B}} \models \text{ZFC}$, then $V_\alpha^{\mathbb{B}} \models \phi$. □

Generic Soundness is immediate for Ω*-logic, and evidently Ω*-logic is the strongest possible logic satisfying this requirement.

The property of generic invariance also holds for Ω*-logic.

Theorem (Generic Invariance). *Suppose that there exists a proper class of Woodin cardinals and that ϕ is a sentence. Then for each complete Boolean algebra $\mathbb{B}$, $\text{ZFC} \vdash_{\Omega^*} \phi$ if and only if*

$$V^{\mathbb{B}} \models \text{“ZFC} \vdash_{\Omega^*} \phi\text{”}. \qquad \square$$

Having defined Ω*-logic, a natural question arises. Is Ω*-logic the same as Ω-logic (at least for Π_2-sentences)? The restriction to Π_2 sentences is a necessary one.

> **Ω Conjecture:** Suppose that there exists a proper class of Woodin cardinals. Then for each Π_2 sentence ϕ, $\text{ZFC} \vdash_{\Omega^*} \phi$ if and only if $\text{ZFC} \vdash_{\Omega} \phi$.

If the Ω Conjecture is true, then I find the argument against CH, based on strong logics, to be a more persuasive one. One reason is that the Ω Conjecture implies that if theory of the structure $\langle H(\omega_2), \in \rangle$ is finitely axiomatized, over ZFC, in Ω*-logic, then CH is *false*.

Connections with the Logic of Large Cardinal Axioms

Ω-logic is intimately connected with an abstract notion of what a large cardinal axiom is. If the Ω Conjecture is true, then the validities of ZFC in Ω-logic—these are the sentences ϕ such that $\text{ZFC} \vdash_\Omega \phi$—calibrate the large cardinal hierarchy.

To illustrate this claim I make the following abstract definition of a large cardinal axiom, essentially identifying large cardinal axioms with one fundamental feature of such axioms. This is the feature of “generic stability”. It is precisely this aspect of large cardinal axioms which underlies the fact that such axioms cannot settle the Continuum Hypothesis. A formula ϕ is a Σ_2 formula if it is of the form $(\exists x_i(\forall x_j \psi))$ where ψ is a Σ_0 formula.

Suppose that κ is an ordinal and that ϕ is a Σ_2 formula. Then “$V \models \phi[\kappa]$” indicates that ϕ is true of κ in V, the universe of sets. Similarly, if $\mathbb{B}$ is a complete Boolean algebra, then “$V^{\mathbb{B}} \models \phi[\kappa]$” indicates that ϕ is true of κ in the Cohen extensions of V that $\mathbb{B}$ could be used to define.

An inaccessible cardinal κ is *strongly inaccessible* if for each cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.

Definition. $(\exists x_1 \phi)$ is *a large cardinal axiom* if $\phi(x_1)$ is a Σ_2-formula; and, as a theorem of ZFC, if κ is a cardinal such that $V \models \phi[\kappa]$, then κ is uncountable, strongly inaccessible, and for all complete Boolean algebras $\mathbb{B}$ of cardinality less than κ, $V^{\mathbb{B}} \models \phi[\kappa]$. □

Definition. Suppose that $(\exists x_1 \phi)$ is a large cardinal axiom. Then V is *ϕ-closed* if for every set X there exist a transitive set M and $\kappa \in M \cap \text{Ord}$ such that

$$\langle M, \in \rangle \models \text{ZFC},$$

$X \in M_\kappa$ and such that $\langle M, \in \rangle \models \phi[\kappa]$. □

The connection between Ω-logic and first order logic is now easily identified.

Lemma. *Suppose that there exists a proper class of Woodin cardinals and that Ψ is a Π_2 sentence. Then $\text{ZFC} \vdash_\Omega \Psi$ if and only if there is a large cardinal axiom $(\exists x_1 \phi)$ such that*

$$\text{ZFC} \vdash_\Omega \text{“}V \text{ is } \phi\text{-closed”}$$

and such that ZFC + “V is ϕ-closed” $\vdash \Psi$.

An immediate corollary of this lemma is that the Ω Conjecture is equivalent to the following conjecture, which actually holds for *all* (conventional) large cardinal axioms currently within reach of the Inner Model Program.

> **Conjecture:** Suppose that there exists a proper class of Woodin cardinals. Suppose that $(\exists x_1 \phi)$ is a large cardinal axiom such that V is ϕ-closed.
>
> Then $\text{ZFC} \vdash_\Omega$ “V is ϕ-closed.”

The equivalence of this conjecture with the Ω Conjecture is essentially a triviality.

Nevertheless, reformulating the Ω Conjecture in this fashion does suggest a route toward proving the Ω Conjecture. Moreover, the reformulation, in conjunction with the preceding lemma, shows quite explicitly that if the Ω Conjecture is true, then Ω-logic is simply the natural logic associated to the set of large cardinal axioms $(\exists x_1 \phi)$ for which V is ϕ-closed.

The Ω Conjecture and the Hierarchy of Large Cardinals

To the uninitiated the plethora of large cardinal axioms seems largely a chaotic collection founded on a wide variety of unrelated intuitions. An enduring mystery of large cardinals is that empirically they really do seem to form a well-ordered hierarchy. The search for an explanation leads to the following question.

> Is it possible to *formally* arrange the large cardinal axioms ($\exists x_1 \phi$) into a well-ordered hierarchy incorporating the known comparisons of specific axioms?

If the Ω Conjecture is true, then the answer is affirmative, at least for those axioms suitably realized within the universe of sets. More precisely, suppose there exists a proper class of Woodin cardinals. The large cardinal axioms ($\exists x_1 \phi$) such that

$$\text{ZFC} \vdash_\Omega \text{ “}V \text{ is } \phi\text{-closed”}$$

are naturally arranged in a well-ordered hierarchy by comparing the minimum possible *lengths* of the Ω-proofs, ZFC $\vdash_\Omega$“V is ϕ-closed”.

If the Ω Conjecture *holds* in V, then this hierarchy includes *all* large cardinal axioms ($\exists x_1 \phi$) such that the universe V is ϕ-closed. This, arguably, accounts for the remarkable success of the view that all large cardinal axioms are comparable. Of course, it is not the large cardinals themselves (κ such that $\phi[\kappa]$ holds) which are directly compared, but the auxiliary notion that the universe is ϕ-closed. Nevertheless, restricted to those large cardinal axioms ($\exists x_1 \phi$) currently within reach of the Inner Model Program, this order coincides with the usual order which is (informally) defined in terms of consistency strength.

Finally, this hierarchy explains, albeit a posteriori, the intertwining of large cardinal axioms and determinacy axioms.

Resolving the Ω Conjecture is essential if we are to advance our understanding of large cardinal axioms. If the Ω Conjecture is true, we obtain, at last, a mathematically precise definition of this hierarchy. But, as one might expect, with this progress come problems (of comparing specific large cardinal axioms) which seem genuinely out of reach of current methods. If the Ω Conjecture is refuted from some large cardinal axiom (which likely must transcend every determinacy axiom), then the explicit hierarchy of large cardinal axioms as calibrated by the validities of Ω-logic is simply an initial segment of something beyond.

Concluding Remarks

So, is the Continuum Hypothesis solvable? Perhaps I am not completely confident the “solution” I have sketched is the solution, but it is for me convincing evidence that there *is* a solution. Thus, I now believe the Continuum Hypothesis is solvable, which is a fundamental change in my view of set theory. While most would agree that a clear resolution of the Continuum Hypothesis would be a remarkable event, it seems relatively few believe that such a resolution will ever happen.

Of course, for the dedicated skeptic there is always the “widget possibility”. This is the future where it is discovered that instead of sets we should be studying widgets. Further, it is realized that the axioms for widgets are obvious and, moreover, that these axioms resolve the Continuum Hypothesis (and everything else). For the eternal skeptic, these widgets are the integers (and the Continuum Hypothesis is resolved as being meaningless).

Widgets aside, the incremental approach sketched in this article comes with a price. What about the general continuum problem; i.e. what about $H(\omega_3)$, $H(\omega_4)$, $H(\omega_{(\omega_1+2010)})$, etc.?

The view that progress towards resolving the Continuum Hypothesis must come with progress on resolving all instances of the Generalized Continuum Hypothesis seems too strong. The understanding of $H(\omega)$ did not come in concert with an understanding of $H(\omega_1)$, and the understanding of $H(\omega_1)$ failed to resolve even the basic mysteries of $H(\omega_2)$. The universe of sets is a large place. We have just barely begun to understand it.

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