Mathematical Logic.

- Proof Theory.
- Recursion Theory.
- Model Theory.
- Set Theory.
Model Theory (1).

Syntax. Symbols Formal Proof ⊢
Semantics. Interpretations Truth ⊨

Gödel’s Completeness Theorem. ⊢ = ⊨ for first order logic.

More precisely: If $T$ is any first-order theory and $\sigma$ any sentence, then the following are equivalent:

1. $T \vdash \sigma$, and
2. for all $M$ such that $M \models T$, we have that $M \models \sigma$. 
Model Theory (2).

If $T$ is any first-order theory and $\sigma$ any sentence, then the following are equivalent:

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2. for all $M$ such that $M \models T$, we have that $M \models \sigma$.

For a set of sentences $T$, let $\text{Mod}(T)$ be the class of models of $T$. For a class of structures $\mathcal{M}$ let $\text{Thy}(\mathcal{M})$ be the class of sentences true in all structures in $\mathcal{M}$.

Then:

$$\text{Thy}(\text{Mod}(T))$$

is the deductive closure of $T$.

It is not true that $\text{Mod}(\text{Thy}(\mathcal{M})) = \mathcal{M}$: let $\mathcal{M} := \{ \mathbb{N} \}$, then there are models $\mathbb{N} \models \text{Th}(\mathbb{N})$ such that $\mathbb{N} \neq \mathbb{N}$. 
Products (1).

Let $\mathcal{L} = \{f_n, R_m ; n, m\}$ be a first-order language and $S$ be a set.

Suppose that for every $i \in S$, we have an $\mathcal{L}$-structure

$$M_i = \langle M_i, f^i_n, R^i_m ; n, m \rangle.$$  

Let $M_S := \prod_{i \in S} M_i$. For $X_0, \ldots, X_k \in M$, we let

$$f^S_n(X_0, \ldots, X_k)(i) := f^i_n(X_0(i), \ldots, X_k(i))$$ 

and

$$R^S_m(X_0, \ldots, X_k) :\iff \forall i \in S(R^i_m(X_0(i), \ldots, X_k(i))).$$
In general, classes of structures are not closed under products:

Let $\mathcal{L}_F := \{ +, \times, 0, 1 \}$ be the language of fields and $\Phi_F$ be the field axioms. Let $S = \{0, 1\}$ and $M_0 = M_1 = \mathbb{Q}$. Then $M_S = \mathbb{Q} \times \mathbb{Q}$ is not a field: $\langle 1, 0 \rangle \in \mathbb{Q} \times \mathbb{Q}$ doesn’t have an inverse.

**Theorem** (Birkhoff, 1935). If a class of algebras is equationally definable, then it is closed under products.

Garrett Birkhoff
(1884-1944)

Ultraproducts (1).

Suppose $S$ is a set, $M_i$ is an $\mathcal{L}$-structure and $U$ is an ultrafilter on $S$.

Define $\equiv_U$ on $M_S$ by

$$X \equiv_U Y :\iff \{ i ; X(i) = Y(i) \} \in U,$$

and let $M_U := M_S/\equiv_U$.

The functions $f_n^S$ and the relations $R_m^S$ are welldefined on $M_U$ (i.e., if $X \equiv_U Y$, then $f_n^S(X) \equiv_U f_n^S(Y)$), and so they induce functions and relations $f_n^U$ and $R_m^U$ on $M_U$.

We call

$$M_U := \text{Ult}(\langle M_i ; i \in S \rangle, U) := \langle M_U, f_n^U, R_m^U ; n, m \rangle$$

the ultraproduct of the sequence $\langle M_i ; i \in S \rangle$ with $U$. 
Theorem (Łoś.) Let $\langle M_i ; i \in S \rangle$ be a family of $\mathcal{L}$-structures and $U$ be an ultrafilter on $S$. Let $\sigma$ be an $\mathcal{L}$-sentence. Then the following are equivalent:

1. $M_U \models \sigma$, and
2. $\{ i \in S ; M_i \models \sigma \} \in U$.

Applications.

- If for all $i \in S$, $M_i$ is a field, then $M_U$ is a field.
- Let $S = \mathbb{N}$. Sets of the form $\{ n ; N \leq n \}$ are called final segments. An ultrafilter $U$ on $\mathbb{N}$ is called nonprincipal if it contains all final segments. If $\langle M_n ; n \in \mathbb{N} \rangle$ is a family of $\mathcal{L}$-structures, $U$ a nonprincipal ultrafilter, and $\Phi$ an (infinite) set of sentences such that each element is “eventually true”, then $M_U \models \Phi$.
- Nonstandard analysis (Robinson). Let $\mathcal{L}$ be the language of fields with an additional 0-ary function symbol $\dot{c}$. Let $M_i \models \text{Th}(\mathbb{R}) \cup \{ \dot{c} \neq 0 \land \dot{c} < \frac{1}{i} \}$. Then $M_U$ is a model of $\text{Th}(\mathbb{R})$ plus “there is an infinitesimal”.

Ultraproducts (2).
Tarski (1).

Alfred Tarski
1902-1983

Teitelbaum (until c. 1923).


1924. Banach-Tarski paradox.

1924-1939. Work in Poland.

1933. *The concept of truth in formalized languages.*

From 1942 at the University of California at Berkeley.

Tarski (2).

**Undefinability of Truth.**
If a language can correctly refer to its own sentences, then the truth predicate is not definable.

**Limitative Theorems.**

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Tarski (2).

- **Undefinability of Truth.**

- **Algebraic Logic.**

  - Leibniz called for an analysis of relations (“Plato is taller than Socrates” $\rightsquigarrow$ “Plato is tall in as much as Socrates is short”).

  - **Relation Algebras:** Steve Givant, István Németi, Hajnal Andréka, Ian Hodkinson, Robin Hirsch, Maarten Marx.

  - **Cylindric Algebras:** Don Monk, Leon Henkin, Ian Hodkinson, Yde Venema, Nick Bezhanishvili.
Tarski (2).

- **Undefinability of Truth.**
- **Algebraic Logic.**
- **Logic and Geometry.**
  - A theory $T$ admits elimination of quantifiers if every first-order formula is $T$-equivalent to a quantifier-free formula (Skolem, 1919).
  - **1955.** Quantifier elimination for the theory of real numbers ("real-closed fields").
  - Basic ideas of modern algebraic model theory.
  - Connections to theoretical computer science: running time of the quantifier elimination algorithms.
Back to Set Theory for a while:

Applications of Set Theory in the foundations of mathematics:
(Remember Hausdorff’s question in pure set theory: are there regular limit cardinals?)

- Vitali’s construction of a non-Lebesgue measurable set (1905).
- Banach’s generalized measure problem (1930): existence of real-valued measurable cardinals.

Banach connects the existence of real-valued measurable cardinals to Hausdorff’s question about inaccessibles:
if Banach’s measure problem has a solution, then Hausdorff’s answer is ‘Yes’.

- Ulam’s notion of a measurable cardinal in terms of ultrafilters.
Early large cardinals.

- Weakly inaccessibles (Hausdorff, 1914).
- Inaccessibles (Zermelo, 1930).
- Real-valued measurables (Banach, 1930).
- Measurables (Ulam, 1930).

Question. Are these notions different? Can we prove that the least inaccessible is not the least measurable?
Recall: A cardinal $\kappa$ is called **measurable** if there is a $\kappa$-complete nonprincipal ultrafilter on $\kappa$.

**Idea:** Apply the theory of ultraproducts to the ultrafilter witnessing measurability.

Let $V$ be a model of set theory and $V \models \text{"}\kappa \text{ is measurable}"$. Let $U$ be the ultrafilter witnessing this. Define $M_\alpha := V$ for all $\alpha \in \kappa$ and $M_U := \text{Ult}(V, U)$.

By Łoś, $M_U$ is again a model of set theory with a measurable cardinal.

**Theorem** (Scott / Tarski-Keisler, 1961). If $\kappa$ is measurable, then there is some $\alpha < \kappa$ such that $\alpha$ is inaccessible.

**Corollary.** The least measurable is not the least inaccessible.
More on large cardinals.

**Reflection.** Some properties of a large cardinal $\kappa$ reflect down to some (many, almost all) cardinals $\alpha < \kappa$.

- **Lévy** (1960); **Montague** (1961). Reflection Principle.
- **Hanf** (1964). Connecting large cardinal analysis to infinitary logic.
- **Gaifman** (1964); **Silver** (1966). Connecting large cardinals and inner models of constructibility (“iterated ultrapowers”).
Hilbert’s First Problem.

Is the Continuum Hypothesis (“every set of reals is either countable or in bijection with the set of all reals”) true?
In 1939, Gödel constructed the constructible universe $L$ and proved:

**Theorem** (Gödel; 1938). $L \models \text{ZFC} + \text{CH}$.

**Corollary.** If $\text{ZF}$ is consistent, then $\text{ZFC} + \text{CH}$ is consistent.

**Consequences.**

- CH cannot be refuted in $\text{ZFC}$.
- The system $\text{ZFC} + \text{CH}$ cannot be logically stronger than $\text{ZF}$, *i.e.*, $\text{ZFC} + \text{CH} \not\vdash \text{Cons(ZF)}$.
- $L$ is tremendously important for the investigation of logical strength. It turns out that if there is a measurable cardinal, then $L \models \text{“there are inaccessible but no measurable cardinals”}$.
- $L$ is a minimal model of set theory.
Gödel’s Constructible Universe (2).

A new axiom? \( V = L \). “The set-theoretic universe is minimal”.

Gödel Rephrased. \( ZF + V = L \vdash AC + CH \).

Possible solutions.

- Prove \( V = L \) from \( ZF \).
- Assume \( V = L \) as an axiom. (\( V = L \) is generally not accepted as an axiom of set theory.)
- Find a different proof of \( AC \) and \( CH \) from \( ZF \).
- Prove \( AC \) and \( CH \) to be independent by creating models of \( ZF + \neg AC \), \( ZF + \neg CH \), and \( ZFC + \neg CH \).
Technique of Forcing (1963). Take a model $M$ of ZFC and a partial order $P \in M$. Then there is a model construction of a new model $M^P$, the forcing extension. By choosing $P$ carefully, we can control properties of $M^P$.

Let $\kappa > \omega_1$. If $P$ is the set of finite partial functions from $\kappa \times \omega$ into $2$, then $M^P \models \neg \text{CH}$.

**Theorem** (Cohen). $\text{ZFC} \not\models \text{CH}$.

**Theorem** (Cohen). $\text{ZF} \not\models \text{AC}$. 
Robert Solovay

- **1962.** Correspondence with Mycielski about the Axiom of Determinacy.

- **1963.** Development of Forcing as a method.

- **1963.** Solves the measure problem: it is consistent with ZF that all sets are Lebesgue measurable.

- **1964.** PhD University of Chicago (advisor: Saunders Mac Lane).

- **1975.** Baker-Gill-Solovay: There are oracles \( p \) and \( q \) such that \( P^p = NP^p \) and \( P^q \neq NP^q \).


- **1977.** Solovay-Strassen algorithm for primality testing.
Now, what is the size of the continuum?

- **Gödel’s Programme.**
  1947. “What is Cantor’s Continuum Problem?”
  Use new axioms (in particular large cardinal axioms) in order to resolve questions undecidable in ZF.

- **Lévy-Solovay** (1967).
  Large Cardinals don’t solve the continuum problem.

More about this in two weeks.
Modal logic (2).

Modalities as operators.
McCull (late XIXth century); Lewis-Langford (1932). ◇ as an operator on propositional expressions:

◇φ → “Possibly φ”.

□ for the dual operator:

□φ → “Necessarily φ”.

Iterated modalities:

□◇φ → “It is necessary that φ is possible”.

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Modal logic (3).

What modal formulas should be axioms? This depends on the interpretation of $\Diamond$ and $\Box$.

**Example.** $\Box \varphi \rightarrow \varphi$ (“axiom $T$”).

- **Necessity interpretation.** “If $\varphi$ is necessarily true, then it is true.”
- **Epistemic interpretation.** “If $p$ knows that $\varphi$, then $\varphi$ is true.”
- **Doxastic interpretation.** “If $p$ believes that $\varphi$, then $\varphi$ is true.”
- **Deontic interpretation.** “If $\varphi$ is obligatory, then $\varphi$ is true.”
Early modal semantics.

**Topological Semantics** (McKinsey / Tarski). Let $\langle X, \tau \rangle$ be a topological space and $V : \mathbb{N} \to \wp(X)$ a valuation for the propositional variables.

$$\langle X, \tau, x, V \rangle \models \Diamond \varphi \text{ if and only if } x \text{ is in the closure of } \{z ; \langle X, \tau, z, V \rangle \models \varphi \}.$$  

$$\langle X, \tau \rangle \models \varphi \text{ if for all } x \in X \text{ and all valuations } V,$$

$$\langle X, \tau, x, V \rangle \models \varphi.$$  

**Theorem** (McKinsey-Tarski; 1944). $\langle X, \tau \rangle \models \varphi$ if and only if $S4 \vdash \varphi$.

$(S4 = \{T, \Box \Box \varphi \to \Box \varphi\})$

"Naming and Necessity".
Kripke semantics (1).

Let $M$ be a set and $R \subseteq M \times M$ a binary relation. We call $M = \langle M, R \rangle$ a Kripke frame. Let $V : \mathbb{N} \rightarrow \wp(M)$ be a valuation function. Then we call $M^V = \langle M, R, V \rangle$ a Kripke model.

\[ M^V, x \models p_n \iff x \in V(n) \]
\[ M^V, x \models \Diamond \varphi \iff \exists y(xRy \land M^V, y \models \varphi) \]
\[ M^V, x \models \Box \varphi \iff \forall y(xRy \rightarrow M^V, y \models \varphi) \]
\[ M^V \models \varphi \iff \forall x(M^V, x \models \varphi) \]
\[ M \models \varphi \iff \forall V(M^V \models \varphi) \]
Kripke semantics (2).

\[ M^V, x \models \Diamond \varphi \iff \exists y (xRy \land M^V, y \models \varphi) \]
\[ M^V, x \models \Box \varphi \iff \forall y (xRy \rightarrow M^V, y \models \varphi) \]

Then \( M \models \varphi \iff \forall V (M^V \models \varphi) \)

Let \( \langle M, R \rangle \) be a reflexive frame, i.e., for all \( x \in M, \ xRx \).
Then \( M \models T \).
\( (T = \Box \varphi \rightarrow \varphi) \)

Let \( \langle M, R \rangle \) be a transitive frame, i.e., for all \( x, y, z \in M, \) if \( xRy \) and \( yRz \), then \( xRz \).
Then \( M \models \Box \Box \varphi \rightarrow \Box \varphi. \)
Theorem (Kripke).

1. $T \models \varphi$ if and only if for all reflexive frames $M$, we have $M \models \varphi$.

2. $S_4 \models \varphi$ if and only if for all reflexive and transitive frames $M$, we have $M \models \varphi$.

3. $S_5 \models \varphi$ if and only if for all frames $M$ with an equivalence relation $R$, we have $M \models \varphi$. 