# Mathematical Logic.

- **Proof Theory.**
- **Recursion Theory.**
- **Model Theory.**
- **Set Theory.**

# Model Theory (1).

| | | | |
|---|---|---|---|
| **Syntax.** | Symbols | Formal Proof | $\vdash$ |
| **Semantics.** | Interpretations | Truth | $\models$ |

**Gödel's Completeness Theorem.** $\vdash \; = \; \models$ for first order logic.

More precisely: If $T$ is any first-order theory and $\sigma$ any sentence, then the following are equivalent:

1. $T \vdash \sigma$, and
2. for all $\mathbf{M}$ such that $\mathbf{M} \models T$, we have that $\mathbf{M} \models \sigma$.

# Model Theory (2).

If $T$ is any first-order theory and $\sigma$ any sentence, then the following are equivalent:

1. $T \vdash \sigma$, and
2. for all $\mathbf{M}$ such that $\mathbf{M} \models T$, we have that $\mathbf{M} \models \sigma$.

For a set of sentences $T$, let $\mathsf{Mod}(T)$ be the class of models of $T$. For a class of structures $\mathcal{M}$ let $\mathsf{Thy}(\mathcal{M})$ be the class of sentences true in all structures in $\mathcal{M}$.

Then:

$$\mathsf{Thy}(\mathsf{Mod}(T)) \text{ is the deductive closure of } T.$$

It is not true that $\mathsf{Mod}(\mathsf{Thy}(\mathcal{M})) = \mathcal{M}$: let $\mathcal{M} := \{\mathbb{N}\}$, then there are models $\mathbf{N} \models \mathrm{Th}(\mathbb{N})$ such that $\mathbf{N} \neq \mathbb{N}$.

# Products (1).

Let $\mathcal{L} = \{\dot{\mathrm{f}}_n, \dot{\mathrm{R}}_m \,;\, n, m\}$ be a first-order language and $S$ be a set.

Suppose that for every $i \in S$, we have an $\mathcal{L}$-structure

$$\mathbf{M}_i = \langle M_i, f_n^i, R_m^i \,;\, n, m\rangle.$$

Let $M_S := \prod_{i\in S} M_i$. For $X_0, ..., X_k \in M$, we let

$$f_n^S(X_0, ..., X_k)(i) := f_n^i(X_0(i), ..., X_k(i)) \text{ and}$$

$$R_m^S(X_0, ..., X_k) :\leftrightarrow \forall i \in S(R_m^i(X_0(i), ..., X_k(i)).$$

# Products (2).

In general, classes of structures are not closed under products:

Let $\mathcal{L}_{\mathrm{F}} := \{+, \times, 0, 1\}$ be the language of fields and $\Phi_{\mathrm{F}}$ be the field axioms. Let $S = \{0, 1\}$ and $\mathbf{M}_0 = \mathbf{M}_1 = \mathbb{Q}$. Then $\mathbf{M}_S = \mathbb{Q} \times \mathbb{Q}$ is not a field: $\langle 1, 0\rangle \in \mathbb{Q} \times \mathbb{Q}$ doesn't have an inverse.
**Theorem** (Birkhoff, 1935). If a class of algebras is equationally definable, then it is closed under products.

Garrett Birkhoff
(1884-1944)

Garrett **Birkhoff**, On the structure of abstract algebras, **Proceedings of the Cambridge Philosophical Society** 31 (1935), p. 433-454

# Ultraproducts (1).

Suppose $S$ is a set, $\mathbf{M}_i$ is an $\mathcal{L}$-structure and $U$ is an ultrafilter on $S$.

Define $\equiv_U$ on $M_S$ by

$$X \equiv_U Y :\leftrightarrow \{i\,;\, X(i) = Y(i)\} \in U,$$

and let $M_U := M_S/{\equiv_U}$.
The functions $f_n^S$ and the relations $R_m^S$ are welldefined on $M_U$ (*i.e.*, if $X \equiv_U Y$, then $f_n^S(X) \equiv_U f_n^S(Y)$), and so they induce functions and relations $f_n^U$ and $R_m^U$ on $M_U$.
We call

$$\mathbf{M}_U := \mathrm{Ult}(\langle \mathbf{M}_i\,;\, i \in S\rangle, U) := \langle M_U, f_n^U, R_m^U\,;\, n, m\rangle$$

the ultraproduct of the sequence $\langle \mathbf{M}_i\,;\, i \in S\rangle$ with $U$.

# Ultraproducts (2).

**Theorem** (Łoś.) Let $\langle \mathbf{M}_i \, ; \, i \in S \rangle$ be a family of $\mathcal{L}$-structures and $U$ be an ultrafilter on $S$. Let $\sigma$ be an $\mathcal{L}$-sentence. Then the following are equivalent:

1. $\mathbf{M}_U \models \sigma$, and
2. $\{i \in S \, ; \, \mathbf{M}_i \models \sigma\} \in U$.

## Applications.

- If for all $i \in S$, $\mathbf{M}_i$ is a fi eld, then $\mathbf{M}_U$ is a fi eld.
- Let $S = \mathbb{N}$. Sets of the form $\{n \, ; \, N \leq n\}$ are called fi nal segments. An ultrafi lter $U$ on $\mathbb{N}$ is called nonprincipal if it contains all fi nal segments. If $\langle \mathbf{M}_n \, ; \, n \in \mathbb{N} \rangle$ is a family of $\mathcal{L}$-structures, $U$ a nonprincipal ultrafi lter, and $\Phi$ an (infi nite) set of sentences such that each element is "eventually true", then $\mathbf{M}_U \models \Phi$.
- **Nonstandard analysis** (Robinson). Let $\mathcal{L}$ be the language of fi elds with an additional $0$-ary function symbol $\dot{c}$. Let $\mathbf{M}_i \models \mathrm{Th}(\mathbb{R}) \cup \{\dot{c} \neq 0 \land \dot{c} < \frac{1}{i}\}$. Then $\mathbf{M}_U$ is a model of $\mathrm{Th}(\mathbb{R})$ plus "there is an infi nitesimal".

# Tarski (1).

Alfred Tarski
1902-1983

- *Teitelbaum* (until c. 1923).
- 1918-1924. Studies in Warsaw. Student of Lesniewski.
- 1924. Banach-Tarski paradox.
- 1924-1939. Work in Poland.
- 1933. *The concept of truth in formalized languages*.
- From 1942 at the University of California at Berkeley.
- **Students. 1946.** Bjarni Jónsson (b. 1920). **1948.** Julia Robinson (1919-1985). **1954.** Bob Vaught (1926-2002). **1957.** Solomon Feferman (b. 1928). **1957.** Richard Montague (1930-1971). **1961.** Jerry Keisler. **1961.** Donald Monk (b. 1930). **1962.** Haim Gaifman. **1963.** William Hanf.

# Tarski (2).

- **Undefinability of Truth.**
  If a language can correctly refer to its own sentences, then the truth predicate is not definable.

  **Limitative Theorems.**

| *Provability* | *Truth* | *Computability* |
|---|---|---|
| 1931 | 1933 | 1935 |
| Gödel | Tarski | Turing |

# Tarski (2).

- **Undefinability of Truth.**
- **Algebraic Logic.**
  - Leibniz called for an analysis of relations ("Plato is taller than Socrates" $\rightsquigarrow$ "Plato is tall in as much as Socrates is short").
  - **Relation Algebras:** Steve Givant, István Németi, Hajnal Andréka, Ian Hodkinson, Robin Hirsch, Maarten Marx.
  - **Cylindric Algebras:** Don Monk, Leon Henkin, Ian Hodkinson, Yde Venema, Nick Bezhanishvili.

# Tarski (2).

- **Undefinability of Truth.**
- **Algebraic Logic.**
- **Logic and Geometry.**
  - A theory $T$ admits elimination of quantifiers if every first-order formula is $T$-equivalent to a quantifier-free formula (Skolem, 1919).
  - **1955.** Quantifier elimination for the theory of real numbers (“real-closed fields”).
  - Basic ideas of modern algebraic model theory.
  - Connections to theoretical computer science: running time of the quantifier elimination algorithms.

# Back to Set Theory for a while:

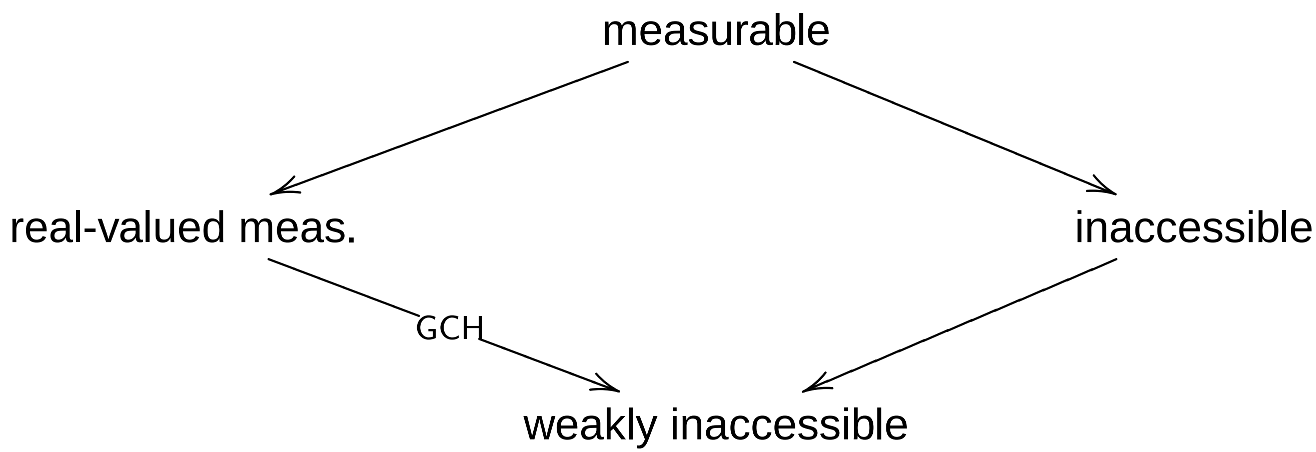
## Applications of Set Theory in the foundations of mathematics:

(Remember Hausdorff's question in pure set theory: are there regular limit cardinals?)

- Vitali's construction of a non-Lebesgue measurable set (1905).
- Hausdorff's Paradox: the Banach-Tarski paradox (1926).
- Banach's generalized measure problem (1930): existence of real-valued measurable cardinals.
- Banach connects the existence of real-valued measurable cardinals to Hausdorff's question about inaccessibles:
  if Banach's measure problem has a solution, then Hausdorff's answer is 'Yes'.
- Ulam's notion of a measurable cardinal in terms of ultrafi lters.

# Early large cardinals.

- Weakly inaccessibles (Hausdorff, 1914).
- Inaccessibles (Zermelo, 1930).
- Real-valued measurables (Banach, 1930).
- Measurables (Ulam, 1930).




**Question.** Are these notions different? Can we prove that the least inaccessible is not the least measurable?

# Ultraproducts in Set Theory.

**Recall:** A cardinal $\kappa$ is called measurable if there is a $\kappa$-complete nonprincipal ultrafilter on $\kappa$.

**Idea:** Apply the theory of ultraproducts to the ultrafilter witnessing measurability.
Let $\mathbf{V}$ be a model of set theory and $\mathbf{V} \models$"$\kappa$ is measurable".
Let $U$ be the ultrafilter witnessing this. Define $\mathrm{M}_\alpha := \mathbf{V}$ for all $\alpha \in \kappa$ and $\mathrm{M}_U := \mathrm{Ult}(\mathbf{V}, U)$.

By Łoś, $\mathrm{M}_U$ is again a model of set theory with a measurable cardinal.

**Theorem** (Scott / Tarski-Keisler, 1961). If $\kappa$ is measurable, then there is some $\alpha < \kappa$ such that $\alpha$ is inaccessible.

**Corollary.** The least measurable is not the least inaccessible.

# More on large cardinals.

**Reflection.** Some properties of a large cardinal $\kappa$ reflect down to some (many, almost all) cardinals $\alpha < \kappa$.

- **Lévy** (1960); **Montague** (1961). Reflection Principle.
- **Hanf** (1964). Connecting large cardinal analysis to infi nitary logic.
- **Gaifman** (1964); **Silver** (1966). Connecting large cardinals and inner models of constructibility (“iterated ultrapowers”).

# Hilbert's First Problem.

Is the Continuum Hypothesis ("every set of reals is either countable or in bijection with the set of all reals") true?

# Gödel's Constructible Universe (1).

In 1939, Gödel constructed the constructible universe $\mathbf{L}$ and proved:

**Theorem** (Gödel; 1938). $\mathbf{L} \models \mathsf{ZFC} + \mathsf{CH}$.

**Corollary.** If $\mathsf{ZF}$ is consistent, then $\mathsf{ZFC} + \mathsf{CH}$ is consistent.

**Consequences.**

- $\mathsf{CH}$ cannot be refuted in $\mathsf{ZFC}$.
- The system $\mathsf{ZFC} + \mathsf{CH}$ cannot be logically stronger than $\mathsf{ZF}$, *i.e.*, $\mathsf{ZFC} + \mathsf{CH} \nvdash \mathrm{Cons}(\mathsf{ZF})$.
- $\mathbf{L}$ is tremendously important for the investigation of logical strength. It turns out that if there is a measurable cardinal, then $\mathbf{L} \models$ "there are inaccessible but no measurable cardinals".
- $\mathbf{L}$ is a minimal model of set theory.

# Gödel's Constructible Universe (2).

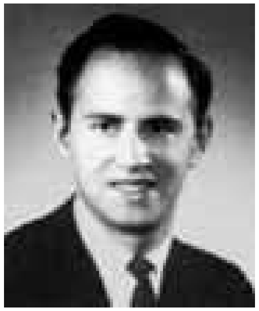
*A new axiom?* $\mathbf{V}{=}\mathbf{L}$. "The set-theoretic universe is minimal".

*Gödel Rephrased.* $\mathsf{ZF} + \mathbf{V}{=}\mathbf{L} \vdash \mathsf{AC} + \mathsf{CH}$.

**Possible solutions.**

- Prove $\mathbf{V}{=}\mathbf{L}$ from $\mathsf{ZF}$.
- Assume $\mathbf{V}{=}\mathbf{L}$ as an axiom. ($\mathbf{V}{=}\mathbf{L}$ is generally not accepted as an axiom of set theory.)
- Find a different proof of $\mathsf{AC}$ and $\mathsf{CH}$ from $\mathsf{ZF}$.
- Prove $\mathsf{AC}$ and $\mathsf{CH}$ to be independent by creating models of $\mathsf{ZF} + \neg\mathsf{AC}$, $\mathsf{ZF} + \neg\mathsf{CH}$, and $\mathsf{ZFC} + \neg\mathsf{CH}$.

# Cohen.

 Paul Cohen (b. 1934)

**Technique of Forcing** (1963). Take a model $M$ of ZFC and a partial order $\mathbb{P} \in M$. Then there is a model construction of a new model $M^{\mathbb{P}}$, the forcing extension. By choosing $\mathbb{P}$ carefully, we can control properties of $M^{\mathbb{P}}$.

Let $\kappa > \omega_1$. If $\mathbb{P}$ is the set of finite partial functions from $\kappa \times \omega$ into $2$, then $M^{\mathbb{P}} \models \neg\mathsf{CH}$.

**Theorem** (Cohen). $\mathsf{ZFC} \nvdash \mathsf{CH}$.

**Theorem** (Cohen). $\mathsf{ZF} \nvdash \mathsf{AC}$.

# Solovay.

## Robert Solovay

- **1962**. Correspondence with Mycielski about the Axiom of Determinacy.
- **1963**. Development of Forcing as a method.
- **1963**. Solves the measure problem: it is consistent with ZF that al sets are Lebesgue measurable.
- **1964**. PhD University of Chicago (advisor: Saunders Mac Lane).
- **1975**. Baker-Gill-Solovay: There are oracles $p$ and $q$ such that $\mathbf{P}^p = \mathbf{NP}^p$ and $\mathbf{P}^q \neq \mathbf{NP}^q$.
- **1976**. Solovay-Woodin: Solution of the Kaplansky problem in the theory of Banach algebras.
- **1977**. Solovay-Strassen algorithm for primality testing.

# Now, what is the size of the continuum?

- **Gödel's Programme.**
  **1947.** "What is Cantor's Continuum Problem?"

  Use new axioms (in particular large cardinal axioms) in order to resolve questions undecidable in ZF.
- **Lévy-Solovay** (1967).
  Large Cardinals don't solve the continuum problem.

More about this in two weeks.

# Modal logic (2).

**Modalities as operators.**

McColl (late XIXth century); Lewis-Langford (1932). $\Diamond$ as an operator on propositional expressions:

$$\Diamond\varphi \rightsquigarrow \text{“Possibly } \varphi\text{”.}$$

$\Box$ for the dual operator:

$$\Box\varphi \rightsquigarrow \text{“Necessarily } \varphi\text{”.}$$

Iterated modalities:

$$\Box\Diamond\varphi \rightsquigarrow \text{“It is necessary that } \varphi \text{ is possible”.}$$

# Modal logic (3).

What modal formulas should be axioms? This depends on the interpretation of $\diamondsuit$ and $\Box$.
**Example.** $\Box\varphi \to \varphi$ (“axiom $\mathbf{T}$”).

- *Necessity interpretation.* “If $\varphi$ is necessarily true, then it is true.”
- *Epistemic interpretation.* “If $p$ knows that $\varphi$, then $\varphi$ is true.”
- *Doxastic interpretation.* “If $p$ believes that $\varphi$, then $\varphi$ is true.”
- *Deontic interpretation.* “If $\varphi$ is obligatory, then $\varphi$ is true.”

# Early modal semantics.

**Topological Semantics** (McKinsey / Tarski).
Let $\langle X, \tau \rangle$ be a topological space and $V : \mathbb{N} \to \wp(X)$ a valuation for the propositional variables.

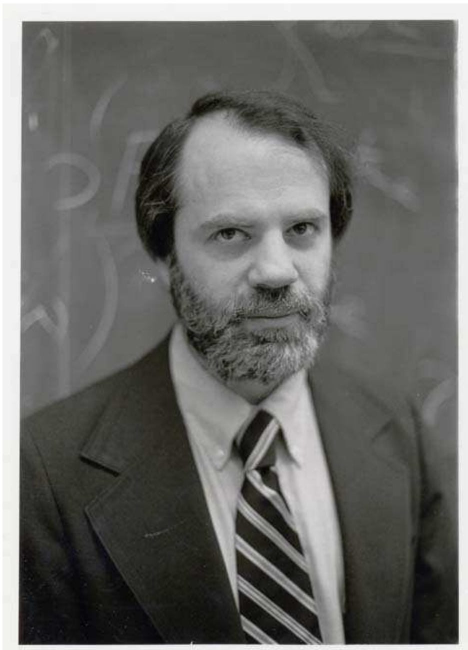
$\langle X, \tau, x, V \rangle \models \Diamond\varphi$ if and only if $x$ is in the closure of $\{z\,;\,\langle X, \tau, z, V \rangle \models \varphi\}$.

$\langle X, \tau \rangle \models \varphi$ if for all $x \in X$ and all valuations $V$, $\langle X, \tau, x, V \rangle \models \varphi$.

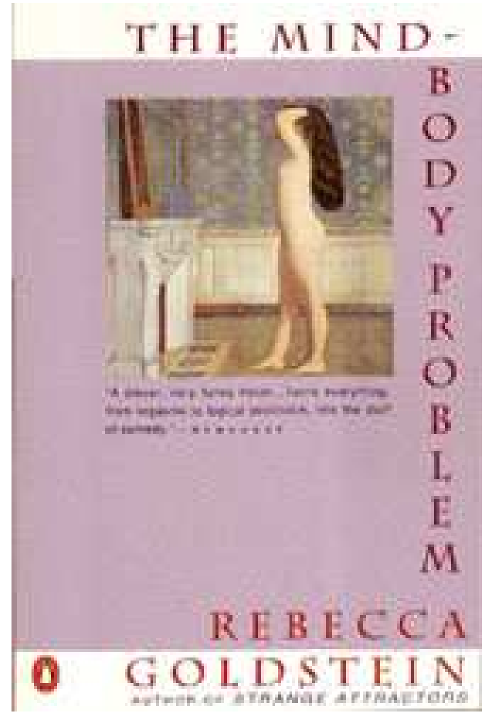
**Theorem** (McKinsey-Tarski; 1944). $\langle X, \tau \rangle \models \varphi$ if and only if $\mathbf{S4} \vdash \varphi$.

($\mathbf{S4} = \{\mathbf{T}, \Box\Box\varphi \to \Box\varphi\}$)

# Kripke.



Saul Kripke
(b. 1940)




- Saul **Kripke**, A completeness theorem in modal logic, **Journal of Symbolic Logic** 24 (1959), p. 1-14.
- "*Naming and Necessity*".

# Kripke semantics (1).

Let $M$ be a set and $R \subseteq M \times M$ a binary relation. We call $\mathbf{M} = \langle M, R \rangle$ a Kripke frame. Let $V : \mathbb{N} \to \wp(M)$ be a valuation function. Then we call $\mathbf{M}^V = \langle M, R, V \rangle$ a Kripke model.

$$
\begin{array}{rcl}
\mathbf{M}^V, x \models \mathrm{p}_n & \text{iff} & x \in V(n) \\
\mathbf{M}^V, x \models \Diamond\varphi & \text{iff} & \exists y(xRy \;\&\; \mathbf{M}^V, y \models \varphi) \\
\mathbf{M}^V, x \models \Box\varphi & \text{iff} & \forall y(xRy \to \mathbf{M}^V, y \models \varphi) \\
\mathbf{M}^V \models \varphi & \text{iff} & \forall x(\mathbf{M}^V, x \models \varphi) \\
\mathbf{M} \models \varphi & \text{iff} & \forall V(\mathbf{M}^V \models \varphi)
\end{array}
$$

# Kripke semantics (2).

$$
\begin{array}{rcl}
\mathbf{M}^V, x \models \Diamond\varphi & \text{iff} & \exists y(xRy \;\&\; \mathbf{M}^V, y \models \varphi) \\
\mathbf{M}^V, x \models \Box\varphi & \text{iff} & \forall y(xRy \rightarrow \mathbf{M}^V, y \models \varphi) \\
\mathbf{M}^V \models \varphi & \text{iff} & \forall x(\mathbf{M}^V, x \models \varphi) \\
\mathbf{M} \models \varphi & \text{iff} & \forall V(\mathbf{M}^V \models \varphi)
\end{array}
$$

- Let $\langle M, R \rangle$ be a reflexive frame, *i.e.*, for all $x \in M$, $xRx$. Then $\mathbf{M} \models \mathbf{T}$. ($\mathbf{T} = \Box\varphi \rightarrow \varphi$)
- Let $\langle M, R \rangle$ be a transitive frame, *i.e.*, for all $x, y, z \in M$, if $xRy$ and $yRz$, then $xRz$. Then $\mathbf{M} \models \Box\Box\varphi \rightarrow \Box\varphi$.

# Kripke semantics (3).

**Theorem** (Kripke).

1. $\mathbf{T} \vdash \varphi$ if and only if for all reflexive frames $\mathbf{M}$, we have $\mathbf{M} \models \varphi$.
2. $\mathbf{S4} \vdash \varphi$ if and only if for all reflexive and transitive frames $\mathbf{M}$, we have $\mathbf{M} \models \varphi$.
3. $\mathbf{S5} \vdash \varphi$ if and only if for all frames $\mathbf{M}$ with an equivalence relation $R$, we have $\mathbf{M} \models \varphi$.