UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

# Core Logic

**2005/2006; 1st Semester**
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**Homework Set # 8** *Deadline:* November 8th, 2005

**Exercise 25** (total of six points).

Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ be a Boolean algebra. Define an operation $\star$ by $x \star y := -(x + y)$ (the NOR or Sheffer operation).

(1) Give formulas $\varphi_{\rm mult}$, $\varphi_{\rm add}$, $\varphi_{\rm comp}$ in the language just containing $\star$, $=$ and parentheses such that

$$\begin{aligned} \varphi_{\rm mult}(x,y,z) &\equiv x \cdot y = z \\ \varphi_{\rm add}(x,y,z) &\equiv x + y = z \\ \varphi_{\rm comp}(x,z) &\equiv -x = z \end{aligned}$$

(1 point each). (In other words, the $\star$-language is expressive enough to define the language of Boolean algebras.)

(2) Prove that the following three so-called "Sheffer axioms" hold for $\star$ (1 point each):

$$(x \star x) \star (x \star x) = x$$

$$x \star (y \star (y \star y)) = x \star x$$

$$(x \star (y \star z)) \star (x \star (y \star z)) = ((y \star y) \star x) \star ((z \star z) \star x)$$

**Exercise 26** (total of three points).

Give the names of the following people (1 point each):

- $X$ was a Aristotelian philosopher from Constantinople who lived in Italy most of his life. From 1456 to 1458, he was the professor for rhetoric and poetics at the *studio fiorentino* and one of the teachers of Lorenzo de'Medici (*il Magnifico*).
- $Y$ was one of the authors of *La logique, ou l'art de penser*. He was called "the Great" to distinguish him from his father who had the same name.
- $Z$ was a niece of King Frederick the Great of Prussia. She was the recipient of the *Lettres à une Princesse d'Allemande* in which Euler explained deductive reasoning by what we now call "Euler diagrams".

**Exercise 27** (total of nine points).

A structure $\langle R, +, \cdot, 0, 1 \rangle$ is called a **ring** if $+$ is commutative and associative binary operation on $R$, $\cdot$ is an associative binary operation on $R$, $\cdot$ distributes over $+$ (*i.e.*, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$), $0$ is the neutral element of $+$ (*i.e.*, $0 + a = a + 0 = a$) and $1$ is the neutral element of $\cdot$ (*i.e.*, $a \cdot 1 = 1 \cdot a = a$).

Examples of rings are: the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$.

Let $\mathbf{B} = \langle B, 0, 1, \vee, \wedge, - \rangle$ be a Boolean algebra. For $X, Y \in B$, define

$$X + Y := (X \wedge -Y) \vee (-X \wedge Y), \text{ and}$$

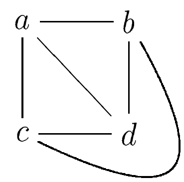
$$X \cdot Y := X \wedge Y.$$

We write $\mathrm{R}(\mathbf{B}) := \langle B, +, \cdot, 0, 1\rangle$.

(1) Prove that $\mathrm{R}(\mathbf{B})$ is a ring (3 points).
(2) Give an example of a ring $R$ such that $R$ is not isomorphic to any $\mathrm{R}(\mathbf{B})$ (with a proof; 4 points).
(3) Prove that there cannot be any three-element Boolean algebra (2 points).

**Exercise 28** (total of four points; three extra points).

Consider the following plane geometry:




Let $P = \{a, b, c, d\}$, $L = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ and $pI\ell$ if $p \in \ell$. Show that $\mathbf{P} := \langle P, L, I\rangle$ is a strongly Euclidean plane (4 points).

*For students with a mathematical background:* What does this example have to do with the two-dimensional vector space over the field $\mathbb{Z}/(2)$ (1 extra point)? Can you come up with an analogous example for the two-dimensional vector space over the field $\mathbb{Z}/(3)$ (2 extra points)?