Deductivism,

and David Hilbert's Grundlagen

David Hilbert

“Wir müssen wissen, wir werden wissen."

* 23rd January 1863 in Königsberg (Prussia)

† 14th Februari 1943 in Göttingen (Germany)

Brief Biography (1)

- 1863 – 1943
- Attended University of Königsberg
- Doctorate in 1885
- Thesis: Über invariante Eigenschaften specieller binärer Formen, insbedondere der Kugelfunctionen
- In 1892 he married Käthe Jerosch had one son: Franz
- 1893 Became Professor

Brief Biography (2)

- **Felix Klein** wanted Hilbert at the Göttingen chair and succeeded in 1895.
- 1899 Grundlagen der Geometrie.
- 1900 Address to the **Second International Congress of Mathematicians** in Paris
- The two wars

Personality (…2)

- Otto Blumenthal (First Student)

In the analysis of mathematical talent one has to differentiate between the ability to create new concepts that generate new types of thought structures and the gift for sensing deeper connections and underlying unity.

In Hilbert's case, his greatness lies in an immensely powerful insight that penetrates into the depths of a question. **All of his works contain examples from far-flung fields in which only he was able to discern an interrelatedness and connection with the problem at hand.** From these, the synthesis, his work of art,was ultimately created.

Game Formalism

Main philosophical problems of GF:

- Truth
- Applicability

Mental Exercise:

- Take a system of axioms and rules
- Suppose we can find an environment where we can “interpret” the axioms and they turn out to be “true”
- Are we justified to regard the theorems as being “true”?

… and Deductivism

- Take the suppose to the extreme. We don’t look for an environment but stick to the “suppose”.
- **If-Then**-ism
- We have stripped mathematics of all content beyond logic. The meaning of mathematics is its logical content.

So..

- What is mathematics about?
- What is mathematical knowledge?
- What about applicability?
- Why?

Grundlagen (1899)

- Hilbert turned to the project of stripping geometry of intuition

- In a retrospective encyclopaedia article, Hilbert's student and protégé Paul Bernays states:

A main feature of Hilbert's axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has generally been adopted in modern mathematics.

…It consists in abstracting from the intuitive meaning of the terms ... and in understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true **for any interpretation ... for which the axioms are satisfied**. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure ...

As Hilbert put it:

...in a proper axiomatization of geometry, one must always be able to say, instead of "points, straight lines, and planes", "tables, chairs, and beer mugs"...

- **the axioms of geometry do not tie the subject to our intuitive notions of points lines planes.**

…quote continued…

…On this conception of axiomatics, ... logical reasoning on the basis of the axioms is used not merely as a means of assisting intuition in the study of spatial figures; rather **logical dependencies are considered for their own sake**…

- the proper subject of geometry is study of the consequences of the axioms, not that of geometric intuition.
- This is true for mathematics in general.

Undefined terms

- Points
- Lines
- Planes
- Lie on, contains
- Between
- Congruent

Axioms (1)

- ***Axioms of Incidence***

- **Postulate I.1.**

For every two points *A*, *B* there exists a line *a* that contains each of the points *A*, *B*.

- **Postulate I.2.**

For every two points *A*, *B* there exists no more than one line that contains each of the points *A*, *B*.

[etc]

Axioms (2)

- ***Axioms of Order***

- **Postulate II.1.**

If a point *B* lies between a point *A* and a point *C* then the points *A*, *B*, *C* are three distinct points of a line, and *B* then also lies between *C* and *A*.

- **Postulate II.2.**

For two points *A* and *C*, there always exists at least one point *B* on the line *AC* such that *C* lies between *A* and *B*.

[etc]

Observations

- Axioms define a relational structure of undefined terms
- Anything at all can play the role of the undefined terms
- Axioms as definitions of primitive principles (instead of attempting to define the "undefined terms")

Hilberts “programme”

- Hilbert constructed a model showing that axioms can be interpreted as true at the same time. (**satisfiability**)
- Gave models where one of the axioms was false but the other true (**independence**).
- Both activities show distrust for the role of intuition.All this is done on a purely logical basis, without appeal to intuition.
- In fact the deductive system is formulated with such clarity that it can itself be studied as a mathematical object

Frege vs. Hilbert (1)

- Frege was a platonist. No if-then-ism in sight.
- Hilbert regarded rejection of content as a major **innovation:**
- [A]ny theory can always be applied to infinitely many systems of basic elements. One only needs to apply a reversible one-one transformation and lay it down that the axioms shall be correspondingly the same for the transformed things. This circumstance is in fact frequently made use of, *e.g.*, in the principle of duality ... [This] ... **can never be a defect in a theory, and it is in any case unavoidable**.

Frege vs. Hilbert (2)

- Frege: definitions provide meaning, axioms provide truth.

axioms must not contain a word or sign whose sense and meaning, or whose contribution to the expression of a thought, was not already completely laid down.

.. the meanings of the words ‘point’, ‘line’, ‘between’ are not given, but are assumed to be known in advance ... [I]t is also left unclear what you call a point. One first thinks of points in the sense of Euclidean geometry, ... But afterwards you think of a pair of numbers as a point ... Here the axioms are made to carry a burden that belongs to definitions ... [B]eside the old meaning of the word ‘axiom’, ... there emerges another meaning but one which I cannot grasp.

Role of axioms

- Hilbert: Axioms as definitions of primitive principles (instead of attempting to define the “undefined terms”)
- definitions cannot be given (at least not in a mathematical way). Axioms provide merely “ifs”.
- Problem with deductivism: role of **Metamathematics**

The 2nd Problem

- Are the axioms of arithmetic consistent?
- To provide a foundation for mathematics; prove the logical consistency of the basis of mathematics
- Can it be done?
- Gödel's incompleteness theorem indicated that the answer is "no," in the sense that any formal system interesting enough to formulate its own consistency can prove its own consistency iff it is inconsistent.